# **Probability Theory 2 :** Solution Sheet 8

## Exercice 1

Let $t > s \geq 0$ and $\Phi(x, y) = x + y$. Then, $\Phi$ is measurable. Since $B_t - B_s$ and $\mathcal{F}_s$ are independents, using the hint yields

$$\begin{aligned}\mathbb{P}\{B_t \in A \mid \mathcal{F}_s\} &= \mathbb{P}\{B_t - B_s + B_s \in A \mid \mathcal{F}_s\} \\ &= \mathbb{P}\{\Phi(B_t - B_s, B_s) \in A \mid \mathcal{F}_s\} \\ &= \mathbb{P}\{\Phi(B_t - B_s, B_s) \in A \mid B_s\} \\ &= \mathbb{P}\{B_t \in A \mid B_s\}.\end{aligned}$$

Denote

$$Q_{t-s}(x, A) := \mathbb{P}\{B_t \in A \mid B_s = x\}.$$

Notice that this notation make sense because

$$\mathbb{P}\{B_t \in A \mid B_s = x\} = \mathbb{P}\{B_{t-s} \in A \mid B_0 = x\},$$

i.e. this probability depend only on $t - s$. It represent the probability that $B_t$ reach $A$ in time $t - s$ given that $B_s = x$. We have to prove that $(Q_t)$ is a transition kernel family, i.e. that Kolmogorov-Chapmann equation hold. Let $x \in \mathbb{R}$ and $A$ measurable.

$$\begin{aligned}Q_{t+u}(x, A) &= \mathbb{P}\{B_{t+u+s} \in A \mid B_s = x\} \\ &= \int_{\mathbb{R}} \mathbb{P}\{B_{t+u+s} \in A \mid B_{u+s} = y, B_s = x\}\mathbb{P}\{B_{u+s} \in \mathrm{d}y \mid B_s = x\} \\ &\underset{(1)}{=} \int_{\mathbb{R}} \mathbb{P}\{B_{t+u+s} \in A \mid B_{u+s} = y\}\mathbb{P}\{B_{u+s} \in \mathrm{d}y \mid B_s = x\} \\ &= \int_{\mathbb{R}} Q_t(y, A) Q_u(x, \mathrm{d}y),\end{aligned}$$

where we used Markov property in (1). Therefore, $(Q_t)$ is a transition kernel family.

We have that

$$Q_t(x, A) = \mathbb{P}\{B_t \in A \mid B_0 = x\} = \int_A f_{B_t|B_0=x}(y)\,\mathrm{d}y = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{(x-y)^2}{2t}}\,\mathrm{d}y,$$

as wished.

## Exercice 2

We use the following theorem :

**Théorème 0.1.**

*Let $\varphi : I \times J \to \mathbb{R}$ where $\varphi = \varphi(x, t)$ a measurable function s.t.*

1. *The function $x \mapsto f(x, t)$ is $L^1$ for all $t \in J$,*
2. *For a.e. $x \in I$, the function $t \mapsto \varphi(x, t)$ is differentiable on $J$,*
3. *There is a function $\kappa : J \to \mathbb{R}$ that is $L^1$ s.t. $|\partial_t \varphi(x, t)| \leq \kappa(x)$ for a.e. $x \in I$. Then $t \mapsto \int_I \varphi(x, t)\,\mathrm{d}x$, is differentiable and*

$$\partial_t \int_I \varphi(x, t)\,\mathrm{d}x = \int_I \partial_t \varphi(x, t)\,\mathrm{d}x,$$

*for all $t \in J$.*

Set $\varphi(x,t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ and let $M > 0$ s.t. $|f(x)| \le M$ for all $x \in \mathbb{R}$.

- We have that

$$g(x,t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x+u) e^{-\frac{u^2}{2t}} \, \mathrm{d}u = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(u) e^{-\frac{(x-u)^2}{2t}} \, \mathrm{d}u.$$

  Let $0 < \delta_1 < t < \delta_2$. Then

$$\left| \partial_t f(x+u) \frac{1}{\sqrt{2\pi t e^{-\frac{u^2}{2t}}}} \right| = |f(x+u)| \left| \frac{-1}{2\sqrt{2\pi} t^{3/2}} + \frac{(x-y)^2}{2\sqrt{2\pi} t^{5/2}} \right| e^{-\frac{u^2}{2t}}$$
$$\le M \left( \frac{1}{2\sqrt{2\pi}\delta_1^{3/2}} + \frac{(x-y)^2}{2\sqrt{2\pi}\delta_1^{5/2}} \right) e^{-\frac{u^2}{2\delta_2}} \in L^1(\mathbb{R}).$$

  Therefore, by Theorem 0.1,

$$\partial_t g(x,t) = \int_{\mathbb{R}} f(x+u) \partial_t \varphi(u,t) \, \mathrm{d}u = \int_{\mathbb{R}} f(u) \partial_t \varphi(x-u,t) \, \mathrm{d}u, \tag{1}$$

  for all $t \in [\delta_1, \delta_2]$. Since $\delta_1$ and $\delta_2$ are unspecified, (1) hold for all $t > 0$. We follow that same strategy for $\partial_x g(x,t)$. Let $\delta_1 < x < \delta_2$. We can suppose WLOG that $x > 0$ and $\delta_1 > 0$ (the strategy when $\delta_1 < x < \delta_2$ when $\delta_1 < 0$ and $\delta_2 > 0$ is exactly the same)

$$\left| \partial_x e^{-\frac{(x-u)^2}{2t}} \right| = |x-y| e^{-\frac{(x-u)^2}{2t}} \le (|y| + \delta_2) e^{-\frac{\delta_1^2}{2t}} e^{\frac{-u^2}{2t}} e^{-u\delta_1} \in L^1(\mathbb{R}).$$

  and thus, by Theorem 0.1,

$$\partial_x g(x,t) = \int_{\mathbb{R}} f(u) \partial_x \varphi(x-u,t) \, \mathrm{d}u = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u,t) \, \mathrm{d}u, \tag{2}$$

  for $x \in [\delta_1, \delta_2]$. Since $\delta_1, \delta_2 \in \mathbb{R}$ are unspecified, (2) holds for all $x \in \mathbb{R}$. The proof that

$$\partial_{xx} g(x,t) = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u,t) \, \mathrm{d}u = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u,t) \, \mathrm{d}u,$$

  goes through the same. One can easily prove that

$$\partial_t \varphi(x-u,t) - \frac{1}{2} \partial_{xx} \varphi(x-u,t) = 0,$$

  and thus, by has been made before, $g$ solve the Heat equation.

- Since $f$ is bounded, we can use DCT what gives

$$\lim_{t \to 0^+} g(x,t) = \mathbb{E}[\lim_{t \to 0} f(x+B_t)] \underset{(1)}{=} \mathbb{E}[f(x+B_0)] = f(x),$$

  where (1) comes from continuity of $f$ and $t \mapsto B_t$.

## Exercice 3

**1.** Suppose $x > 0$.

$$\mathbb{P}\{\tau_x \ge t\} \underset{(1)}{=} \mathbb{P}\left\{ \sup_{s \in [0,t]} B_s \le x \right\} = 1 - 2\mathbb{P}\{B_s \ge x\} \underset{(2)}{=} \sqrt{\frac{2}{\pi t}} \int_0^x e^{-\frac{x^2}{2t}} \, \mathrm{d}x,$$

where we used reflection principle in (1) and in (2), we made the calculation in exercise 3 of sheet 7. If $x < 0$, then

$$\mathbb{P}\{\tau_x \ge t\} = \mathbb{P}\left\{ \inf_{s \in [0,t]} B_s \ge x \right\} = 1 - \mathbb{P}\left\{ \sup_{s \in [0,t]} (-B_s) \ge -x \right\},$$

and the proof claim follow as previously.

**2.** Let $0 < s < t < \infty$.

$$\begin{aligned}
\mathbb{P}\{\forall u \in (s,t), B_u \neq 0\} &= \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (s,t), B_u \neq 0 \mid B_s = x\}\mathbb{P}\{B_s \in \,\mathrm{d}x\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (0,t-s), B_{u+s} \neq 0 \mid B_s = x\}\mathbb{P}\{B_s \in \,\mathrm{d}x\} \\
&\underset{(3)}{=} \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (0,t-s), B_u \neq -x \mid B_0 = 0\}\mathbb{P}\{B_s \in \,\mathrm{d}x\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{\tau_{-x} > t-s \mid B_0 = 0\}\mathbb{P}\{B_s \in \,\mathrm{d}x\},
\end{aligned}$$

We have to justify (3) properly. Let $\{t_n\}_{n\in\mathbb{N}}$ an enumeration of $(0,t-s)\cap\mathbb{Q}$.

$$\begin{aligned}
\mathbb{P}\{\forall u \in (0,t-s), B_{u+s} \neq 0 \mid B_s = x\} &\underset{(4)}{=} \mathbb{P}\left(\bigcap_{u\in(0,t-s)\cap\mathbb{Q}} \{B_{u+s} \neq 0\} \mid B_s = x\right) \\
&= \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=0}^{n}\{B_{t_i+s} \neq 0\} \mid B_s = x\right). \\
&\underset{(5)}{=} \lim_{n\to\infty} \mathbb{P}\left(\bigcap_{i=0}^{n}\{B_{t_i+s} \neq -x\} \mid B_0 = 0\right) \\
&\underset{(6)}{=} \mathbb{P}\left\{\forall u \in (0,t-s), B_u \neq -x \mid B_0 = 0\right\}.
\end{aligned}$$

(4) follow from the continuity of Brownian motion. For (5), remark that if $t_1 < t_2 < t_3$, using Markov property $(a)$, time homogeneity $(b)$, invariance by translation $(c)$ yields

$$\begin{aligned}
\mathbb{P}\{B_{t_3} \leq x_3, B_{t_2} \leq x_2 \mid B_{t_1} = x_1\} &\underset{(a)}{=} \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3} \leq x_3 \mid B_{t_2} = \alpha\}\mathbb{P}\{B_{t_2} \in \,\mathrm{d}\alpha \mid B_{t_1} = x_1\} \\
&\underset{(b)}{=} \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \alpha\}\mathbb{P}\{B_{t_2-t_1} \in \,\mathrm{d}\alpha \mid B_0 = x_1\} \\
&\underset{(c)}{=} \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \alpha\}\mathbb{P}\{B_{t_2-t_1} \in \,\mathrm{d}\alpha - x_1 \mid B_0 = 0\} \\
&\underset{\beta=\alpha-x_1}{=} \int_{-\infty}^{x_2-x_1} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \beta + x_1\}\mathbb{P}\{B_{t_2-t_1} \in \,\mathrm{d}\beta \mid B_0 = 0\} \\
&\underset{(c)}{=} \int_{-\infty}^{x_2-x_1} \mathbb{P}\{B_{t_3-t_1} \leq x_3 - x_1 \mid B_{t_2-t_1} = \beta\}\mathbb{P}\{B_{t_2-t_1} \in \,\mathrm{d}\beta \mid B_0 = 0\} \\
&= \mathbb{P}\{B_{t_3-t_1} \leq x_3 - x_1, B_{t_2-t_1} \leq x_2 - x_1 \mid B_0 = 0\}.
\end{aligned}$$

Then, (5) follow by induction. Finally, (6) follow by the continuity of the probability and the continuity of the the Brownian motion.

**3.** Combine **1.** and **2.** yields

$$\begin{aligned}
h(s) &= 2\int_0^\infty \sqrt{\frac{2}{\pi(t-s)}} \int_0^b e^{-\frac{x^2}{2(t-s)}} \,\mathrm{d}x \frac{1}{\sqrt{2\pi s}} e^{-\frac{b^2}{2s}} \,\mathrm{d}b \\
&= \frac{2}{\pi} \int_0^\infty \int_0^{b\sqrt{\frac{s}{t-s}}} e^{-\frac{x^2}{2}} \,\mathrm{d}x e^{-\frac{b^2}{2}} \,\mathrm{d}b.
\end{aligned}$$

Let $\delta_1, \delta_2 > 0$ s.t. $0 < \delta_1 < s < \delta_2 < t$. If

$$g(s) = \int_0^{b\sqrt{\frac{s}{t-s}}} e^{-\frac{x^2}{2}} \,\mathrm{d}x,$$

then
$$\left|g'(s)e^{-\frac{b^2}{2}}\right| = \frac{t}{t-s}\cdot\frac{b}{\sqrt{s(t-s)}}e^{-\frac{b^2 s}{t-s}} \leq \frac{t}{t-\delta_2}\frac{1}{\sqrt{\delta_1(t-\delta_2)}}e^{-\frac{b^2\delta_2}{t-\delta_1}}e^{-\frac{b^2}{2}} \in L^1.$$
Therefore, using Theorem 0.1 yield
$$h'(s) = \frac{2}{\pi}\int_0^\infty \frac{t}{t-s}\cdot\frac{1}{\sqrt{s(t-s)}}e^{--b^2\cdot\frac{s}{t-s}}e^{-\frac{b^2}{2}}\,\mathrm{d}b = \frac{1}{\pi}\cdot\frac{1}{\sqrt{s(t-s)}}, \tag{3}$$
for all $s\in(\delta_1,\delta_2)$. Since $\delta_1,\delta_2>0$ are unspecified, (3) hold for all $0<s<t$. Since $h(0)=0$, integrating yields
$$h(s) = \frac{2}{\pi}\arcsin\left(\sqrt{\frac{s}{t}}\right),$$
as wished.

## Exercice 4

**1.** We have
$$\sum_{i=0}^{n-1} B_{t_{i+1}}(B_{t_{i+1}}-B_{t_i}) = \sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}}-B_{t_i}) + \sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})^2.$$
By the lecture, we know that
$$\sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}}-B_{t_i}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t B_t\,\mathrm{d}B_t \quad\text{and}\quad \sum_{i=0}^{n-1}(B_{t_{i+1}}-B_{t_i})^2 \xrightarrow[n\to\infty]{\mathbb{P}} t,$$
where $Y_n \xrightarrow[n\to\infty]{\mathbb{P}} Y$ mean that $(Y_n)$ convergence to $Y$ in probability. Therefore
$$\sum_{i=0}^{n-1} B_{t_{i+1}}(B_{t_{i+1}}-B_{t_i}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t B_s\,\mathrm{d}B_s + t.$$

**2.** As previously
$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}}-B_{t_i}) = \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}}-B_{t_i}) + \sum_{i=0}^{n-1}(X_{t_{i+1}}-X_{t_i})(B_{t_{i+1}}-B_{t_i}).$$
Since $X$ has a.s. finite variation path,
$$\begin{aligned}\left|\sum_{i=0}^{n-1}(X_{t_{i+1}}-X_{t_i})(B_{t_{i+1}}-B_{t_i})\right| &\leq \sup_{i=0,\dots,n-1}|B_{t_{i+1}}-B_{t_i}|\sum_{i=0}^{n-1}|X_{t_{i+1}}-X_{t_i}| \\ &\leq C\sup_{i=0,\dots,n-1}|B_{t_{i+1}}-B_{t_i}| \xrightarrow[n\to\infty]{} 0 \quad \text{a.s.}\end{aligned}$$
Therefore
$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}}-B_{t_i}) \xrightarrow[n\to\infty]{\mathbb{P}} \int_0^t X_s\,\mathrm{d}B_s.$$

**3.** When $X$ has a.s. finite variation path, then
$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}}-B_{t_i}) \quad\text{and}\quad \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}}-B_{t_i}),$$
has the same limit (in probability), which fails when $X$ has not a.s. finite variation path.