

Probability Theory 2 : Solution Sheet 8

Exercise 1

Let $t > s \geq 0$ and $\Phi(x, y) = x + y$. Then, Φ is measurable. Since $B_t - B_s$ and \mathcal{F}_s are independents, using the hint yields

$$\begin{aligned} \mathbb{P}\{B_t \in A \mid \mathcal{F}_s\} &= \mathbb{P}\{B_t - B_s + B_s \in A \mid \mathcal{F}_s\} \\ &= \mathbb{P}\{\Phi(B_t - B_s, B_s) \in A \mid \mathcal{F}_s\} \\ &= \mathbb{P}\{\Phi(B_t - B_s, B_s) \in A \mid B_s\} \\ &= \mathbb{P}\{B_t \in A \mid B_s\}. \end{aligned}$$

Denote

$$Q_{t-s}(x, A) := \mathbb{P}\{B_t \in A \mid B_s = x\}.$$

Notice that this notation make sense because

$$\mathbb{P}\{B_t \in A \mid B_s = x\} = \mathbb{P}\{B_{t-s} \in A \mid B_0 = x\},$$

i.e. this probability depend only on $t - s$. It represent the probability that B_t reach A in time $t - s$ given that $B_s = x$. We have to prove that (Q_t) is a transition kernel family, i.e. that Kolmogorov-Chapmann equation hold. Let $x \in \mathbb{R}$ and A measurable.

$$\begin{aligned} Q_{t+u}(x, A) &= \mathbb{P}\{B_{t+u+s} \in A \mid B_s = x\} \\ &= \int_{\mathbb{R}} \mathbb{P}\{B_{t+u+s} \in A \mid B_{u+s} = y, B_s = x\} \mathbb{P}\{B_{u+s} \in dy \mid B_s = x\} \\ &\stackrel{(1)}{=} \int_{\mathbb{R}} \mathbb{P}\{B_{t+u+s} \in A \mid B_{u+s} = y\} \mathbb{P}\{B_{u+s} \in dy \mid B_s = x\} \\ &= \int_{\mathbb{R}} Q_t(y, A) Q_u(x, dy), \end{aligned}$$

where we used Markov property in (1). Therefore, (Q_t) is a transition kernel family.

We have that

$$Q_t(x, A) = \mathbb{P}\{B_t \in A \mid B_0 = x\} = \int_A f_{B_t|B_0=x}(y) dy = \frac{1}{\sqrt{2\pi t}} \int_A e^{-\frac{(x-y)^2}{2t}} dy,$$

as wished.

Exercise 2

We use the following theorem :

Théorème 0.1.

Let $\varphi : I \times J \rightarrow \mathbb{R}$ where $\varphi = \varphi(x, t)$ a measurable function s.t.

1. The function $x \mapsto \varphi(x, t)$ is L^1 for all $t \in J$,
2. For a.e. $x \in I$, the function $t \mapsto \varphi(x, t)$ is differentiable on J ,
3. There is a function $\kappa : J \rightarrow \mathbb{R}$ that is L^1 s.t. $|\partial_t \varphi(x, t)| \leq \kappa(x)$ for a.e. $x \in I$. Then $t \mapsto \int_I \varphi(x, t) dx$, is differentiable and

$$\partial_t \int_I \varphi(x, t) dx = \int_I \partial_t \varphi(x, t) dx,$$

for all $t \in J$.

Set $\varphi(x, t) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}$ and let $M > 0$ s.t. $|f(x)| \leq M$ for all $x \in \mathbb{R}$.

- We have that

$$g(x, t) = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(x+u) e^{-\frac{u^2}{2t}} du = \frac{1}{\sqrt{2\pi t}} \int_{\mathbb{R}} f(u) e^{-\frac{(x-u)^2}{2t}} du.$$

Let $0 < \delta_1 < t < \delta_2$. Then

$$\begin{aligned} \left| \partial_t f(x+u) \frac{1}{\sqrt{2\pi t} e^{-\frac{u^2}{2t}}} \right| &= |f(x+u)| \left| \frac{-1}{2\sqrt{2\pi t^{3/2}} + \frac{(x-y)^2}{2\sqrt{2\pi t^{5/2}}}} \right| e^{-\frac{u^2}{2t}} \\ &\leq M \left(\frac{1}{2\sqrt{2\pi} \delta_1^{3/2}} + \frac{(x-y)^2}{2\sqrt{2\pi} \delta_1^{5/2}} \right) e^{-\frac{u^2}{2\delta_2}} \in L^1(\mathbb{R}). \end{aligned}$$

Therefore, by Theorem 0.1,

$$\partial_t g(x, t) = \int_{\mathbb{R}} f(x+u) \partial_t \varphi(u, t) du = \int_{\mathbb{R}} f(u) \partial_t \varphi(x-u, t) du, \quad (1)$$

for all $t \in [\delta_1, \delta_2]$. Since δ_1 and δ_2 are unspecified, (1) hold for all $t > 0$. We follow that same strategy for $\partial_x g(x, t)$. Let $\delta_1 < x < \delta_2$. We can suppose WLOG that $x > 0$ and $\delta_1 > 0$ (the strategy when $\delta_1 < x < \delta_2$ when $\delta_1 < 0$ and $\delta_2 > 0$ is exactly the same)

$$\left| \partial_x e^{-\frac{(x-u)^2}{2t}} \right| = |x-y| e^{-\frac{(x-u)^2}{2t}} \leq (|y| + \delta_2) e^{-\frac{\delta_1^2}{2t}} e^{-\frac{u^2}{2t}} e^{-u\delta_1} \in L^1(\mathbb{R}).$$

and thus, by Theorem 0.1,

$$\partial_x g(x, t) = \int_{\mathbb{R}} f(u) \partial_x \varphi(x-u, t) du = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u, t) du, \quad (2)$$

for $x \in [\delta_1, \delta_2]$. Since $\delta_1, \delta_2 \in \mathbb{R}$ are unspecified, (2) holds for all $x \in \mathbb{R}$. The proof that

$$\partial_{xx} g(x, t) = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u, t) du = \int_{\mathbb{R}} f(u) \partial_{xx} \varphi(x-u, t) du,$$

goes through the same. One can easily prove that

$$\partial_t \varphi(x-u, t) - \frac{1}{2} \partial_{xx} \varphi(x-u, t) = 0,$$

and thus, by has been made before, g solve the Heat equation.

- Since f is bounded, we can use DCT what gives

$$\lim_{t \rightarrow 0^+} g(x, t) = \mathbb{E}[\lim_{t \rightarrow 0} f(x+B_t)] \stackrel{(1)}{=} \mathbb{E}[f(x+B_0)] = f(x),$$

where (1) comes from continuity of f and $t \mapsto B_t$.

Exercise 3

1. Suppose $x > 0$.

$$\mathbb{P}\{\tau_x \geq t\} \stackrel{(1)}{=} \mathbb{P}\left\{ \sup_{s \in [0, t]} B_s \leq x \right\} = 1 - 2\mathbb{P}\{B_s \geq x\} \stackrel{(2)}{=} \sqrt{\frac{2}{\pi t}} \int_0^x e^{-\frac{x^2}{2t}} dx,$$

where we used reflection principle in (1) and in (2), we made the calculation in exercise 3 of sheet 7. If $x < 0$, then

$$\mathbb{P}\{\tau_x \geq t\} = \mathbb{P}\left\{ \inf_{s \in [0, t]} B_s \geq x \right\} = 1 - \mathbb{P}\left\{ \sup_{s \in [0, t]} (-B_s) \geq -x \right\},$$

and the proof claim follow as previously.

2. Let $0 < s < t < \infty$.

$$\begin{aligned}
\mathbb{P}\{\forall u \in (s, t), B_u \neq 0\} &= \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (s, t), B_u \neq 0 \mid B_s = x\} \mathbb{P}\{B_s \in dx\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (0, t-s), B_{u+s} \neq 0 \mid B_s = x\} \mathbb{P}\{B_s \in dx\} \\
&= \int_{\mathbb{R}} \mathbb{P}\{\forall u \in (0, t-s), B_u \neq -x \mid B_0 = 0\} \mathbb{P}\{B_s \in dx\} \\
&\stackrel{(3)}{=} \int_{\mathbb{R}} \mathbb{P}\{\tau_{-x} > t-s \mid B_0 = 0\} \mathbb{P}\{B_s \in dx\},
\end{aligned}$$

We have to justify (3) properly. Let $\{t_n\}_{n \in \mathbb{N}}$ an enumeration of $(0, t-s) \cap \mathbb{Q}$.

$$\begin{aligned}
\mathbb{P}\{\forall u \in (0, t-s), B_{u+s} \neq 0 \mid B_s = x\} &\stackrel{(4)}{=} \mathbb{P}\left(\bigcap_{u \in (0, t-s) \cap \mathbb{Q}} \{B_{u+s} \neq 0\} \mid B_s = x\right) \\
&= \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{i=0}^n \{B_{t_i+s} \neq 0\} \mid B_s = x\right). \\
&\stackrel{(5)}{=} \lim_{n \rightarrow \infty} \mathbb{P}\left(\bigcap_{i=0}^n \{B_{t_i} \neq -x\} \mid B_0 = 0\right) \\
&\stackrel{(6)}{=} \mathbb{P}\{\forall u \in (0, t-s), B_u \neq -x \mid B_0 = 0\}.
\end{aligned}$$

(4) follow from the continuity of Brownian motion. For (5), remark that if $t_1 < t_2 < t_3$, using Markov property (a), time homogeneity (b), invariance by translation (c) yields

$$\begin{aligned}
\mathbb{P}\{B_{t_3} \leq x_3, B_{t_2} \leq x_2 \mid B_{t_1} = x_1\} &\stackrel{(a)}{=} \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3} \leq x_3 \mid B_{t_2} = \alpha\} \mathbb{P}\{B_{t_2} \in d\alpha \mid B_{t_1} = x_1\} \\
&= \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \alpha\} \mathbb{P}\{B_{t_2-t_1} \in d\alpha \mid B_0 = x_1\} \\
&\stackrel{(b)}{=} \int_{-\infty}^{x_2} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \alpha\} \mathbb{P}\{B_{t_2-t_1} \in d\alpha - x_1 \mid B_0 = 0\} \\
&\stackrel{(c)}{=} \int_{-\infty}^{x_2-x_1} \mathbb{P}\{B_{t_3-t_1} \leq x_3 \mid B_{t_2-t_1} = \beta + x_1\} \mathbb{P}\{B_{t_2-t_1} \in d\beta \mid B_0 = 0\} \\
&= \int_{-\infty}^{x_2-x_1} \mathbb{P}\{B_{t_3-t_1} \leq x_3 - x_1 \mid B_{t_2-t_1} = \beta\} \mathbb{P}\{B_{t_2-t_1} \in d\beta \mid B_0 = 0\} \\
&\stackrel{(c)}{=} \int_{-\infty}^{x_2-x_1} \mathbb{P}\{B_{t_3-t_1} \leq x_3 - x_1, B_{t_2-t_1} \leq x_2 - x_1 \mid B_0 = 0\}.
\end{aligned}$$

Then, (5) follow by induction. Finally, (6) follow by the continuity of the probability and the continuity of the the Brownian motion.

3. Combine 1. and 2. yields

$$\begin{aligned}
h(s) &= 2 \int_0^\infty \sqrt{\frac{2}{\pi(t-s)}} \int_0^b e^{-\frac{x^2}{2(t-s)}} dx \frac{1}{\sqrt{2\pi s}} e^{-\frac{b^2}{2s}} db \\
&= \frac{2}{\pi} \int_0^\infty \int_0^{b\sqrt{\frac{s}{t-s}}} e^{-\frac{x^2}{2}} dx e^{-\frac{b^2}{2}} db.
\end{aligned}$$

Let $\delta_1, \delta_2 > 0$ s.t. $0 < \delta_1 < s < \delta_2 < t$. If

$$g(s) = \int_0^{b\sqrt{\frac{s}{t-s}}} e^{-\frac{x^2}{2}} dx,$$

then

$$\left| g'(s)e^{-\frac{b^2}{2}} \right| = \frac{t}{t-s} \cdot \frac{b}{\sqrt{s(t-s)}} e^{-\frac{b^2 s}{t-s}} \leq \frac{t}{t-\delta_2} \frac{1}{\sqrt{\delta_1(t-\delta_2)}} e^{-\frac{b^2 \delta_2}{t-\delta_1}} e^{-\frac{b^2}{2}} \in L^1.$$

Therefore, using Theorem 0.1 yield

$$h'(s) = \frac{2}{\pi} \int_0^\infty \frac{t}{t-s} \cdot \frac{1}{\sqrt{s(t-s)}} e^{-b^2 \cdot \frac{s}{t-s}} e^{-\frac{b^2}{2}} db = \frac{1}{\pi} \cdot \frac{1}{\sqrt{s(t-s)}}, \quad (3)$$

for all $s \in (\delta_1, \delta_2)$. Since $\delta_1, \delta_2 > 0$ are unspecified, (3) hold for all $0 < s < t$. Since $h(0) = 0$, integrating yields

$$h(s) = \frac{2}{\pi} \arcsin \left(\sqrt{\frac{s}{t}} \right),$$

as wished.

Exercise 4

1. We have

$$\sum_{i=0}^{n-1} B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) = \sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) + \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2.$$

By the lecture, we know that

$$\sum_{i=0}^{n-1} B_{t_i}(B_{t_{i+1}} - B_{t_i}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t B_t dB_t \quad \text{and} \quad \sum_{i=0}^{n-1} (B_{t_{i+1}} - B_{t_i})^2 \xrightarrow[n \rightarrow \infty]{\mathbb{P}} t,$$

where $Y_n \xrightarrow[n \rightarrow \infty]{\mathbb{P}} Y$ mean that (Y_n) convergence to Y in probability. Therefore

$$\sum_{i=0}^{n-1} B_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t B_s dB_s + t.$$

2. As previously

$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) = \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i}) + \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(B_{t_{i+1}} - B_{t_i}).$$

Since X has a.s. finite variation path,

$$\begin{aligned} \left| \sum_{i=0}^{n-1} (X_{t_{i+1}} - X_{t_i})(B_{t_{i+1}} - B_{t_i}) \right| &\leq \sup_{i=0, \dots, n-1} |B_{t_{i+1}} - B_{t_i}| \sum_{i=0}^{n-1} |X_{t_{i+1}} - X_{t_i}| \\ &\leq C \sup_{i=0, \dots, n-1} |B_{t_{i+1}} - B_{t_i}| \xrightarrow[n \rightarrow \infty]{} 0 \quad \text{a.s.} \end{aligned}$$

Therefore

$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) \xrightarrow[n \rightarrow \infty]{\mathbb{P}} \int_0^t X_s dB_s.$$

3. When X has a.s. finite variation path, then

$$\sum_{i=0}^{n-1} X_{t_{i+1}}(B_{t_{i+1}} - B_{t_i}) \quad \text{and} \quad \sum_{i=0}^{n-1} X_{t_i}(B_{t_{i+1}} - B_{t_i}),$$

has the same limit (in probability), which fails when X has not a.s. finite variation path.