Probability Theory 2 : Solution Sheet 7

Exercice 1

Let $\{t_n\}_{n\in\mathbb{N}^*}$ an enumeration of $[0,\infty)\cap\mathbb{Q}$. Set $D_n=\{t_1,\ldots,t_n\}$ and $D=\bigcup_{n\in\mathbb{N}^*}D_n$.

1. By Doob inequality on D_n, we have

$$\lambda^p\mathbb{P}\left\{\sup_{1\le k\le n}|X_{t_k}|\ge\lambda\right\}\le\sup_{1\le k\le n}\mathbb{E}[|X_{t_k}|^p].$$

Since

$$\sup_{1\le k\le n}|X_{t_k}|\nearrow\sup_{t\in D}|X_t|\quad\text{a.s.}$$

using monotone convergence theorem and

$$\sup_{1\le k\le n}\mathbb{E}[|X_{t_k}|^p]\nearrow\sup_{t\in D}\mathbb{E}|X_t|,$$

gives

$$\lambda^p\mathbb{P}\left\{\sup_{t\in D}|X_t|\ge\lambda\right\}\le\sup_{t\in D}\mathbb{E}[|X_t|^p].$$

Since (X_t) is right continuous, we have

$$\sup_{t\in D}|X_t|=\sup_{t\ge 0}|X_t|,$$

what prove the claim.

2. The proof is similar and is left to the reader.

Exercice 2

1. Let X a $\mathcal{A}-$measurable r.v. We have to prove that

$$\mathbb{E}[e^{t_1X+t_2Y}]=\mathbb{E}[e^{t_1X}]\mathbb{E}[e^{t_2Y}].$$

But this is rather clear since

$$\mathbb{E}[e^{t_1X}e^{t_2Y}]=\mathbb{E}\big[e^{t_1X}\mathbb{E}[e^{t_2Y}\mid\mathcal{A}]\big]=\mathbb{E}[e^{t_1X}]\mathbb{E}[e^{t_2Y}],$$

where the first equality come from the fact that e^{t_1X} is $\mathcal{A}-$measurable.

2. $\boxed{\Rightarrow}$ Suppose (M_t^α) is a Martingale for all $\alpha\in\mathbb{R}$. Let $0\le s<t$. Then

$$\begin{aligned}\mathbb{E}[e^{\alpha(X_t-X_s)}]&=\mathbb{E}\left[e^{\alpha(X_t-X_s)-\frac{\alpha^2}{2}(t-s)}\right]\\&=e^{\frac{\alpha^2}{2}(t-s)}\mathbb{E}\left[e^{\alpha(X_t-X_s)-\frac{\alpha^2}{2}(t-s)}\right]\\&\underset{(1)}{=}e^{\frac{\alpha^2}{2}(t-s)}\mathbb{E}\left[e^{-\alpha X_s+\frac{\alpha^2}{2}s}\mathbb{E}\left[e^{\alpha X_t-\frac{\alpha^2}{2}t}\mid\mathcal{F}_s\right]\right]\\&\underset{(2)}{=}e^{\frac{\alpha^2}{2}(t-s)}\end{aligned}$$

where we used the fact that $e^{\alpha X_s+\frac{\alpha^2}{2}s}$ is $\mathcal{F}_s$ measurable in (1), and the martingale property in (2). Therefore $X_t-X_s\sim\mathcal{N}(0,t-s)$. Moreover, for all $\alpha\in\mathbb{R}$,

$$\mathbb{E}[e^{\alpha(X_t-X_s)}\mid\mathcal{F}_s]=e^{-\alpha X_s+\frac{\alpha^2}{2}t}\mathbb{E}\left[e^{\alpha X_t-\frac{\alpha^2}{2}t}\mid\mathcal{F}_s\right]=e^{\frac{\alpha^2}{2}(t-s)}=\mathbb{E}[e^{\alpha(X_t-X_s)}],$$

and thus, by question **1.**, X_t-X_s and $\mathcal{F}_s$ are independents. Since $X_0=0$ a.s. and is continuous, we conclude that (X_t) is a Brownian motion.

$\boxed{\Leftarrow}$ Suppose (X_t) is a Brownian motion. The fact that $\mathbb{E}\left[\left|e^{\alpha X_t - \frac{\alpha^2}{2}t}\right|\right] < \infty$ for all $t \geq 0$ and that $e^{\alpha X_t - \frac{\alpha^2}{2}t}$ is $\mathcal{F}_t := \sigma(X_s \mid s \leq t)$ adapted for all $t \geq 0$ are clear. Let $\alpha \in \mathbb{R}$ and $0 \leq s < t$.

$$\begin{aligned}\mathbb{E}\left[e^{\alpha X_t - \frac{\alpha^2}{2}t} \mid \mathcal{F}_s\right] &\underset{(3)}{=} e^{\alpha X_s - \frac{\alpha^2}{2}t}\mathbb{E}\left[e^{\alpha(X_t - X_s)} \mid \mathcal{F}_s\right] \\ &\underset{(4)}{=} e^{\alpha X_s - \frac{\alpha^2}{2}t}\mathbb{E}\left[e^{\alpha(X_t - X_s)}\right] \\ &\underset{(5)}{=} e^{\alpha X_s - \frac{\alpha^2}{2}s},\end{aligned}$$

where we used the fact that $e^{\alpha X_s}$ is $\mathcal{F}_s$ measurable in (3), that $X_t - X_s$ is independent of $\mathcal{F}_s$ in (4) and that $X_t - X_s \sim \mathcal{N}(0, t-s)$ in (5). Therefore, $\left(e^{\alpha X_t - \frac{\alpha^2}{2}t}\right)$ is a martingale for all $\alpha \in \mathbb{R}$.

Exercice 3

1. Using reflexion principle yields

$$\begin{aligned}\mathbb{P}\{\tau \geq T\} &= \mathbb{P}\left\{\sup_{t\in[0,T]} B_t \leq a\right\} \\ &= 1 - 2\mathbb{P}\{B_T > a\} \\ &= 1 - \frac{2}{\sqrt{2\pi T}}\int_{-\infty}^{-a} e^{-\frac{x^2}{2T}}\,\mathrm{d}x \\ &= \frac{1}{\sqrt{2\pi T}}\int_{-a}^{a} e^{-\frac{x^2}{2T}}\,\mathrm{d}x.\end{aligned}$$

Since

$$\int_{-a}^{a} e^{-\frac{x^2}{2T}}\,\mathrm{d}x = \frac{1}{2}\int_0^a e^{\frac{-x^2}{2T}}\,\mathrm{d}x \geq \frac{a}{2}e^{-\frac{a^2}{2T}},$$

we get

$$\mathbb{E}[\tau] = \int_0^\infty \mathbb{P}\{\tau \geq T\}\,\mathrm{d}T = \infty.$$

2. Set

$$M_t = e^{\sqrt{2\lambda}B_t - \lambda t}.$$

By the previous exercise, (M_t) is a martingale. Then so is $(M_{\tau\wedge n})_{n\in\mathbb{N}}$. By the optional stopping theorem,

$$\mathbb{E}[M_{\tau\wedge n}] = \mathbb{E}[M_0] = 1.$$

Since

$$|M_{\tau\wedge n}| = e^{\sqrt{2\lambda}B_{\tau\wedge n} - \lambda(\tau\wedge n)} \leq e^{\sqrt{2\lambda}a},$$

we have by DCT (dominated convergence theorem) that

$$\mathbb{E}[M_\tau] = \lim_{n\to\infty}\mathbb{E}[M_{\tau\wedge n}] = 1.$$

Therefore

$$\mathbb{E}[e^{\sqrt{2\lambda}a - \lambda\tau}] = 1,$$

and thus

$$\mathbb{E}[e^{-\lambda\tau}] = e^{-2\lambda a}.$$