Probability Theory 2 : Solution Sheet 6

Recall that $a^+ := a \vee 0$ where $\alpha \vee \beta$ denotes the maximum between α and β. It's not used here, but we also define $a^- := -(a \wedge 0)$ where $\alpha \wedge \beta$ denotes the minimum between α and β. So, as you can see a^+ and a^- are always positives and

$$a = a^+ - a^- \quad \text{and} \quad |a| = a^+ + a^-.$$

These relations are not so important here, but they are often used in books, articles...

Exercice 1

I want to precise that this exercise is really not so easy, so don't panic if you didn't solve it properly.

Let $\{t_n\}_n$ an enumeration of $D := [a, b) \cap \mathbb{Q}$. Set $D_n = \{t_1, \ldots, t_n\}$. We denote $U(D_n, [\alpha, \beta])$ the number of upcrossing through $[\alpha, \beta]$ of $(X_t)_{t \in D_n}$.

• **Claim 1 :** $U(D, [\alpha, \beta]) := \sup_{n \in \mathbb{N}^*} U(D_n, [\alpha, \beta])$ is a.s. finite.
By the Doob's upcrossing estimate,

$$\mathbb{E}[U(D_n, [\alpha, \beta])] \leq \frac{1}{\beta - \alpha} \mathbb{E}[(X_{t_n} - \alpha)^+].$$

Since $x \mapsto x^+$ is convex, $\big((X_t - \alpha)^+\big)$ is a submartingale, and thus,

$$\mathbb{E}[(X_{t_n} - \alpha)^+] \leq \mathbb{E}[(X_t - \alpha)^+],$$

for all $t > t_n$. Moreover, since $(X - \alpha)^+ \leq X^+ + |\alpha|$ we get

$$\mathbb{E}[U(D_n, [\alpha, \beta])] \leq \frac{\sup_{t \geq 0} \mathbb{E}[X_t^+] + |\alpha|}{\beta - \alpha} < \infty.$$

Since $D_n \nearrow D$ we have $U(D_n, [\alpha, \beta]) \nearrow U(D, [\alpha, \beta])$. Therefore MCT (monotone convergence theorem), yields to

$$\mathbb{E}[U(D, [\alpha, \beta])] \leq \frac{\sup_{t \geq 0} \mathbb{E}[X_t^+] + |\alpha|}{\beta - \alpha} < \infty,$$

what prove the claim.

• **Claim 2 :** $\lim_{t \to \infty} X_t$ exist a.s.
By the claim 1, there is a null set $N_{\alpha,\beta}$ s.t.

$$U(D, [\alpha, \beta])(\omega) < \infty \quad \text{for all } \omega \notin N_{\alpha,\beta}.$$

Set $N := \bigcup_{\alpha < \beta, \alpha, \beta \in \mathbb{Q}^+} N_{\alpha,\beta}$. Then, N is a null set. Set $\Omega_0 := \Omega \setminus N$. Then, $\mathbb{P}(\Omega_0) = 1$. By Fatou's lemma, $\liminf_{t \to \infty} X_t$ is finite a.s. Suppose by contradiction that there is $\omega \in \Omega_0$ s.t.

$$\liminf_{t \to \infty} X_t(\omega) < \alpha < \beta < \limsup_{t \to \infty} X_t(\omega),$$

for some rational $0 \leq \alpha < \beta < \infty$. Since (X_t) is right-continuous,

$$\liminf_{\substack{t \to \infty \\ t \in \mathbb{Q}^+}} X_t = \liminf_{t \to \infty} X_t \quad \text{and} \quad \limsup_{t \to \infty} X_t = \limsup_{\substack{t \to \infty \\ t \in \mathbb{Q}^+}} X_t.$$

Therefore, $(X_t(\omega))_{t \in \mathbb{Q}^+}$ has infinitely many upcrossing, which contradict $U(\mathbb{Q}^+, [\alpha, \beta])(\omega) < \infty$. Therefore,

$$\limsup_{t \to \infty} X_t(\omega) = \liminf_{t \to \infty} X_t(\omega) < \infty$$

for all $\omega \in \Omega_0$. Set $X_\infty(\omega) = \lim_{t \to \infty} X_t \mathbf{1}_{\Omega_0}$. Then

$$\lim_{t \to \infty} X_t = X_\infty \quad \text{a.s.}$$

Exercice 2

- **1.⇒2.** : Since (X_t) is uniformly integrable, $\sup_t \mathbb{E}[|X_t|] < \infty$. Therefore, by exercise 1, there is X a.s. finite s.t.
$$\lim_{t\to\infty} X_t = X \quad \text{a.s.}$$
Moreover, $X \in L^1$ since if (t_n) is a sequence s.t. $t_n \to \infty$, by Fatou's lemma
$$\mathbb{E}[|X|] \leq \liminf_{n\to\infty} \mathbb{E}[|X_{t_n}|] \leq \sup_{t\geq 0} \mathbb{E}[|X_t|] < \infty.$$
By exercise 3 of sheet 1, $X_{t_n} \to X$ in L^1 for all sequences (t_n) s.t. $t_n \to \infty$. Therefore, $X_t \underset{t\to\infty}{\longrightarrow} X$ in L^1.
- **2.⇒ 3.** : Let $X \in L^1$ and suppose $X_t \underset{t\to\infty}{\longrightarrow} X$ in L^1. In particular $\mathbb{E}[X_t] \underset{t\to\infty}{\longrightarrow} \mathbb{E}[X]$. Therefore, if $F \in \mathcal{F}_t$ and $T \geq t$,
$$\mathbb{E}[X\mathbf{1}_F] = \lim_{T\to\infty} \mathbb{E}[X_T\mathbf{1}_F] = \lim_{T\to\infty} \mathbb{E}\big[\mathbb{E}[X_T\mathbf{1}_F \mid \mathcal{F}_t]\big] \underset{(1)}{=} \mathbb{E}[X_t\mathbf{1}_F],$$
where (1) follow from the fact that $F \in \mathcal{F}_t$ and the martingale property, i.e. $\mathbb{E}[X_T \mid \mathcal{F}_t] = X_t$. Since X_t is $\mathcal{F}_t-$measurable, we get $X_t = \mathbb{E}[X \mid \mathcal{F}_t]$, as wished.
- **3.⇒1.** : *We have made the proof in the first sheet, exercise 3. The proof is exactly the same. I write it anyway, so that you have a nice version.*

  Let $\varepsilon > 0$. By continuity of the probability measure, there is $\delta > 0$ s.t.
$$\mathbb{E}[|X|; A] := \int_A |X| \,\mathrm{d}\mathbb{P} \leq \varepsilon, \tag{2}$$
whenever $\mathbb{P}(A) \leq \delta$. Take M big enough to have $\frac{\mathbb{E}[|X|]}{M} \leq \delta$. Then, by Jensen inequality and the fact that $\{|X_t| > M\} \subset \{\mathbb{E}[|Y| \mid \mathcal{F}_t] > M\}$, we have
$$\mathbb{E}[|X_t|; |X_t| > M] \leq \mathbb{E}\big[\mathbb{E}[|Y| \mid \mathcal{F}_t]; \mathbb{E}[|Y| \mid \mathcal{F}_t] > M\big] = \mathbb{E}\big[|Y|; \mathbb{E}[|Y| \mid \mathcal{F}_t] > M\big],$$
where the last equality come from the fact that $\{\mathbb{E}[|Y| \mid \mathcal{F}_t] > M\} \in \mathcal{F}_t$. By Chebychev's inequality and recalling the definition of M yields
$$\mathbb{P}\left\{\mathbb{E}[|Y| \mid \mathcal{F}_t] > M\right\} \leq \frac{\mathbb{E}\big[\mathbb{E}[|Y| \mid \mathcal{F}_t]\big]}{M} = \frac{\mathbb{E}[|X|]}{M} \leq \delta,$$
and thus, by (2),
$$\mathbb{E}\big[|X_t| \mid |X_t| > M\big] \leq \varepsilon,$$
for all t. This prove that
$$\lim_{M\to\infty} \sup_{t\geq 0} \mathbb{E}\big[|X_t|; |X_t| > M\big] = 0.$$