Probability Theory 2 : Solution Sheet 5

We recall that
$$a \wedge b := \min\{a,b\} \quad \text{and} \quad a \vee b = \max\{a,b\}.$$

Exercice 1

Recall that
$$\mathcal{F}_\tau = \{A \in \mathcal{F}_\infty \mid \forall t \geq 0, A \cap \{\tau \leq t\} \in \mathcal{F}_t\},$$
where $\mathcal{F}_\infty = \bigvee_{t\geq 0} \mathcal{F}_t := \sigma\left(\bigcup_{t\geq 0} \mathcal{F}_t\right)$.

1. Since for all $s \geq 0$,
$$\{\tau \leq s\} \cap \{\tau \leq t\} = \{\tau \leq t \wedge s\} \in \mathcal{F}_{t\wedge s} \subset \mathcal{F}_t,$$
τ is $\mathcal{F}_\tau-$measurable.

2. Let $t \geq 0$. Then,
$$\{\sigma \wedge \tau > t\} = \{\sigma > t\} \cap \{\tau > t\} = \{\sigma \leq t\}^c \cap \{\tau \leq t\}^c \in \mathcal{F}_t.$$
Therefore $\sigma \wedge \tau$ is a stopping time. Also,
$$\{\sigma \vee \tau \leq t\} = \{\sigma \leq t\} \cap \{\tau \leq t\} \in \mathcal{F}_t,$$
and thus, $\sigma \vee \tau$ is a stopping time.

3. Let $F \in \mathcal{F}_\sigma$. Since $\sigma \leq \tau$ a.s., we have $\{\tau \leq t\} \subset \{\sigma \leq t\}$. Therefore,
$$F \cap \{\tau \leq t\} = \underbrace{\Big(F \cap \{\sigma \leq t\}\Big)}_{\in \mathcal{F}_t} \cap \underbrace{\{\tau \leq t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t,$$
and thus $F \in \mathcal{F}_\tau$.

4. The inclusion $\mathcal{F}_{\tau\wedge\sigma} \subset \mathcal{F}_\tau \cap \mathcal{F}_\sigma$ directly follow from **2.** and **3.**. For the other inclusion, let $F \in \mathcal{F}_\sigma \cap \mathcal{F}_\tau$. Then,
$$F \cap \{\tau \wedge \sigma \leq t\} = F \cap \Big(\{\tau \leq t\} \cup \{\sigma \leq t\}\Big) = \Big(F \cap \{\sigma \leq t\}\Big) \cup \Big(F \cap \{\tau \leq t\}\Big) \in \mathcal{F}_t,$$
since $F \in \mathcal{F}_\tau \cap \mathcal{F}_\sigma$, what prove the claim.

Exercice 2

$\boxed{\Rightarrow}$ Suppose $X = (X_t)$ is $\mathcal{F}-$adapted where $X_t = \xi \mathbf{1}_{\{\tau \leq t\}}$. Let $t \geq 0$. In particular, for all $B \in \mathcal{B}(\mathbb{R})$,
$$X_t^{-1}(B) \in \mathcal{F}_t.$$
Since $\xi \neq 0$, we have that
$$X_t(\omega) = 0 \iff \tau(\omega) > t,$$
and thus
$$\{\tau > t\} = X_t^{-1}(\{0\}) \in \mathcal{F}_t.$$
Therefore, τ is a stopping time.

Let $B \in \mathcal{B}(\mathbb{R})$. Write $B = A \cup \{0\}$ where $A = B \setminus \{0\}$. Since $\xi^{-1}\{0\} = \emptyset$ and $A \in \mathcal{B}(\mathbb{R})$,
$$\xi^{-1}(B) \cap \{\tau \leq t\} = \xi^{-1}(A) \cap \{\tau \leq t\} = X_t^{-1}(A) \in \mathcal{F}_t,$$
Therefore ξ is $\mathcal{F}_\tau$ measurable.

$\boxed{\Leftarrow}$ Suppose that τ is a stopping time and ξ is $\mathcal{F}_\tau-$measurable. Let $B \in \mathcal{B}(\mathbb{R})$ and write $B = A \cup \{0\}$ where A is defined as previously. Then,
$$X_t^{-1}(B) = \underbrace{\Big(\xi^{-1}(A) \cap \{\tau \leq t\}\Big)}_{\in \mathcal{F}_t} \cup \underbrace{\{\tau > t\}}_{\in \mathcal{F}_t} \in \mathcal{F}_t.$$
Therefore X_t is $\mathcal{F}_t-$measurable and thus X is $\mathcal{F}-$adapted.

Exercice 3

If $\pi_n : 0 = t_0 < \ldots < t^n_{k_n} = t$ is a partition of $[0,t]$, we denote $|\pi_n| := \max_{i=0,\ldots,k_n-1} |t^n_{i+1} - t^n_i|$.

1. Let $\pi_n : 0 \leq t^n_1 < \ldots < t^n_{k_n} = t$ a partition of $[0,t]$ s.t. $|\pi_n| \to 0$. Set $Y^n_i = \left(B_{t^n_{i+1}} - B_{t^n_i}\right)^2$. Let $\varepsilon > 0$ and remark that since $\mathbb{E}[Y^n_i] = t^n_{i+1} - t^n_i$, we have

$$t = \sum_{i=0}^{k_n-1} \mathbb{E}[Y^n_i].$$

Therefore

$$\begin{aligned}
\mathbb{P}\left\{|Q_{\pi_n}(B) - t| > \varepsilon\right\} &= \mathbb{P}\left\{\left|\sum_{i=0}^{k_n-1}\left(Y^n_i - \mathbb{E}[Y^n_i]\right)\right| > \varepsilon\right\} \\
&\underset{(1)}{\leq} \frac{1}{\varepsilon^2}\operatorname{Var}\left(\sum_{i=0}^{k_n-1} Y^n_i\right) \\
&\underset{(2)}{=} \frac{1}{\varepsilon^2}\sum_{i=0}^{k_n-1}\operatorname{Var}(Y^n_i) \\
&\underset{(3)}{=} \frac{2}{\varepsilon^2}\sum_{i=0}^{k_n-1}(t^n_{i+1} - t^n_i)^2 \\
&\leq \frac{2}{\varepsilon}\cdot|\pi_n|\sum_{i=0}^{k_n-1}(t^n_{i+1} - t^n_i) \\
&= \frac{2t}{\varepsilon}\cdot|\pi_n| \underset{n\to\infty}{\longrightarrow} 0,
\end{aligned}$$

where (1) follow from Tchebychev inequality, (2) follow from independence of the Y^n_i for $i \in \{0,\ldots,k_n-1\}$ and (3) come from the fact that $Y^n_i \sim (X^n_i)^2$ where $X^n_i \sim \mathcal{N}(0, t^n_{i+1} - t^n_i)$, and thus

$$\mathbb{E}[Y^n_i] = \mathbb{E}[(X^n_i)^2] = t^n_{i+1} - t_i \quad \text{and} \quad \mathbb{E}[(Y^n_i)^2] = \mathbb{E}[(X^n_i)^4] = 3(t^n_{i+1} - t^n_i)^2.$$

2. Let $\pi_n : 0 = t^n_0 < \ldots < t^n_{k_n} = t$ a partition of $[0,t]$ s.t. $|\pi_n| \to 0$. Since f is continuously differentiable,

$$f(y) - f(x) = \int_x^y f',$$

and thus

$$\sum_{i=0}^{k_n-1}\left(f(t^n_{i+1}) - f(t^n_i)\right)^2 \underset{(4)}{\leq} \sum_{i=0}^{k_n-1}(t^n_{i+1} - t^n_i)\int_{t^n_i}^{t^n_{i+1}} f'(u)^2\,\mathrm{d}u \leq |\pi_n|\int_0^t f'(u)^2\,\mathrm{d}u \underset{n\to\infty}{\overset{(5)}{\longrightarrow}} 0,$$

where we used Cauchy-Schwarz in (4), and (5) follow form the fact that $u \mapsto f'(u)^2$ is continuous on $[0,t]$, and thus integrable on $[0,t]$.

3. By a theorem of the lecture, Brownian motion has quadratic variation $t \neq 0$ on $[0,t]$. Therefore, by question **2.**, it's not continuously differentiable.