Probability Theory 2 : Solution Sheet 4

Exercice 1

1. Since

$$\liminf_{t\to 0^+} \frac{B_t}{\sqrt{2t\log(\log(\frac{1}{t}))}} = -\limsup_{t\to 0^+} \frac{-B_t}{\sqrt{2t\log(\log(\frac{1}{t}))}},$$

and that $(-B_t)$ is a BM, the claim follow from iterated logarithm law.

2. Also,

$$\begin{aligned}\liminf_{s\to +\infty} \frac{B_s}{\sqrt{2s\log(\log(s))}} &\underset{s=\frac{1}{t}}{=} \liminf_{t\to 0^+} \frac{\sqrt{t}B_{\frac{1}{t}}}{\sqrt{2\log(\log(\frac{1}{t}))}} \\ &= -\limsup_{t\to 0^+} \frac{-\sqrt{t}B_{\frac{1}{t}}}{\sqrt{2\log(\log(\frac{1}{t}))}} \\ &= -\limsup_{t\to 0^+} \frac{-tB_{\frac{1}{t}}}{\sqrt{2t\log(\log(\frac{1}{t}))}}.\end{aligned}$$

Since $(-tB_{\frac{1}{t}})$ is a BM, the claim follow from iterated logarithm theorem.

Exercice 2

In this exercise we'll use the following property of normal random variable without justification : if $X \sim \mathcal{N}(\mu, \sigma^2)$, then for all $a > 0$ and all $b \in \mathbb{R}$,

$$X \sim \mathcal{N}(a\mu + b, b^2\sigma^2).$$

The proof is left to the reader as an elementary exercise.

1. X_t is the sum of two a.s. continuous process. Therefore it's continuous. Let $0 \le t_1 < t_2 < \ldots < t_n < \infty$ and $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$.

$$\alpha_1 X_{t_1} + \ldots + \alpha_n X_{t_n} = \sigma \underbrace{\alpha_1 B_{t_1} + \ldots + \alpha_n B_{t_n}}_{:=Y} + \mu \underbrace{(\alpha_1 t_1 + \ldots + \alpha_n t_n)}_{:=\beta}.$$

Since (B_t) is a Gaussian process, Y is a normal random variable and thus so is $\sigma Y + \mu\beta$.

2. We have that

$$X_t \sim \mathcal{N}\big(\mu t, \sigma^2 t\big).$$

We denote $\bar{z}$ the conjugate of z. The characteristic function is defined as

$$\varphi_{X_t}(\xi) := \mathbb{E}[e^{i\xi X_t}].$$

$$\begin{aligned}\overline{\varphi(\xi)} &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i\xi x} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-i\xi x} e^{-\frac{[x-\mu t]^2}{2\sigma^2 t}}\, \mathrm{d}x \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \mathcal{F}\left(x \mapsto e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}}\right)(\xi).\end{aligned}$$

Where

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x)\, \mathrm{d}x,$$

denote the Fourier transform of the function g. By properties of Fourier transform,

$$\mathcal{F}\big(x \mapsto f(x+h)\big) = e^{ih\xi}\mathcal{F}(f)(\xi) \quad \text{and} \quad \mathcal{F}\big(x \mapsto f(ax)\big) = \frac{1}{a}\mathcal{F}(f)\left(\frac{\xi}{a}\right), \quad a > 0.$$

Therefore,

$$\overline{\varphi_{X_t}(\xi)} = \frac{1}{\sqrt{\pi}} e^{i\mu(t-s)\sigma}\mathcal{F}\left(x \mapsto e^{-x^2}\right)\left(\sigma\sqrt{2t}\right)$$

The fact that

$$\mathcal{F}\left(x \mapsto e^{-x^2}\right)(\xi) = \sqrt{\pi}e^{-\frac{\xi^2}{4}},$$

is a standard exercise. Combine all these results yields

$$\varphi_{X_t}(\xi) = e^{i\mu t\xi - \frac{\sigma^2 t\xi^2}{2}}.$$

Remark : If you knew that $\mathbb{E}[e^{i\xi B_t}] = e^{-i\frac{t^2\xi^2}{2}}$, then I accepted answers as

$$\mathbb{E}[e^{i\xi X_t}] = e^{i\mu t\xi}\mathbb{E}[e^{i\xi\sigma B_t}] = e^{i\xi\mu t - \frac{\sigma^2 t\xi^2}{2}},$$

what make the question quite straightforward.

3. Let $t > s$. Remark that (X_t) has stationary increments, i.e. $X_t - X_s \sim X_{t-s}$. Therefore, we immediately have

$$\mathbb{E}\big[e^{X_t - X_s}\big] = \varphi_{X_{t-s}}(-i),$$

and thus,

$$\mathbb{E}\big[e^{X_t - X_s}\big] = e^{\mu(t-s) + \frac{\sigma^2(t-s)}{2}}.$$

Exercice 3

Set $W_t := UB_t$ where U is s.t. $UU^T = \mathbf{1}_{m\times m}$ where $\mathbf{1}_{m\times m}$ denote the identity matrix. The fact that $W_0 = 0$ and that $t \mapsto W_t$ is continuous is clear.

The fact that increments are independents is also clear since if two randoms variables X and Y are independents, then so are $f(X)$ and $f(Y)$ for any measurable function $f : \mathbb{R}^m \to \mathbb{R}^m$. Set $f(x) = Ux$ where $x \in \mathbb{R}^m$. Since $W_t - W_s = f(B_t - B_s)$ for all $t > s$ and that increaments of (B_t) are independents, then so are of increments of (W_t).

Claim : If $t > s$, then $W_t - W_s \sim \mathcal{N}\big(0, (t-s)\mathbf{1}_{m\times m}\big)$.

To prove this statement, we just need to show that

$$\mathbb{E}\big[e^{i\xi^T(W_t - W_s)}\big] = e^{-\frac{1}{2}(t-s)|\xi|^2}.$$

$$\mathbb{E}\left[e^{i\xi^T(W_t - W_s)}\right] = \mathbb{E}\left[e^{i\xi^T U(B_t - B_s)}\right] = \mathbb{E}\left[e^{i(U^T\xi)^T(B_t - B_s)}\right] = e^{-\frac{1}{2}(t-s)|U^T\xi|^2},$$

where the last equality come from the fact that $B_t - B_s \sim \mathcal{N}\big(0, (t-s)\mathbf{1}_{m\times m}\big)$. Since

$$|U^T\xi|^2 = \langle U^T\xi, U^T\xi\rangle = \xi^T UU^T\xi = \xi^T\xi = |\xi|^2,$$

we finally get

$$\mathbb{E}\left[e^{i\xi^T(W_t - W_s)}\right] = e^{-\frac{1}{2}(t-s)|\xi|^2},$$

as wished.

Therefore, (W_t) is a Brownian motion.

Exercice 4

1. Let $0 \leq t_0 < t_1 < \ldots < t_n < \infty$. Obviously,

$$\mathbb{P}\{Y_{t_1} = 0, \ldots, Y_{t_n} = 0\} = 1.$$

Moreover,

$$\mathbb{P}\{X_{t_0} = 0, \ldots, X_{t_n} = 0\} = \mathbb{P}\big([0,\infty) \setminus \{t_0, \ldots, t_n\}\big) = 1,$$

since $\mathbb{P}\{t_i\} = 0$ for all i (because there is no atom).

2. Fix $t \geq 0$.

$$X_t(\omega) = Y_t(\omega) \iff \omega \in [0,\infty) \setminus \{t\}.$$

Therefore,

$$\mathbb{P}\{X_t = Y_t\} = \mathbb{P}\big([0,\infty) \setminus \{t\}\big) = 1.$$

Remark nevertheless that $\mathbb{P}\{\forall t \geq 0, X_t = Y_t\} = 0$, and thus they are not indistinguishable.