

## Probability Theory 2 : Solution Sheet 4

### Exercise 1

1. Since

$$\liminf_{t \rightarrow 0^+} \frac{B_t}{\sqrt{2t \log(\log(\frac{1}{t}))}} = - \limsup_{t \rightarrow 0^+} \frac{-B_t}{\sqrt{2t \log(\log(\frac{1}{t}))}},$$

and that  $(-B_t)$  is a BM, the claim follow from iterated logarithm law.

2. Also,

$$\begin{aligned} \liminf_{s \rightarrow +\infty} \frac{B_s}{\sqrt{2s \log(\log(s))}} &= \liminf_{s=\frac{1}{t}} \frac{\sqrt{t}B_{\frac{1}{t}}}{\sqrt{2 \log(\log(\frac{1}{t}))}} \\ &= - \limsup_{t \rightarrow 0^+} \frac{-\sqrt{t}B_{\frac{1}{t}}}{\sqrt{2 \log(\log(\frac{1}{t}))}} \\ &= - \limsup_{t \rightarrow 0^+} \frac{-tB_{\frac{1}{t}}}{\sqrt{2t \log(\log(\frac{1}{t}))}}. \end{aligned}$$

Since  $(-tB_{\frac{1}{t}})$  is a BM, the claim follow from iterated logarithm theorem.

### Exercise 2

In this exercise we'll use the following property of normal random variable without justification :  $X \sim \mathcal{N}(\mu, \sigma^2)$ , then for all  $a > 0$  and all  $b \in \mathbb{R}$ ,

$$X \sim \mathcal{N}(a\mu + b, b^2\sigma^2).$$

The proof is left to the reader as an elementary exercise.

1.  $X_t$  is the sum of two a.s. continuous process. Therefore it's continuous. Let  $0 \leq t_1 < t_2 < \dots < t_n < \infty$  and  $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ .

$$\alpha_1 X_{t_1} + \dots + \alpha_n X_{t_n} = \sigma \underbrace{\alpha_1 B_{t_1} + \dots + \alpha_n B_{t_n}}_{:=Y} + \mu \underbrace{(\alpha_1 t_1 + \dots + \alpha_n t_n)}_{:=\beta}.$$

Since  $(B_t)$  is a Gaussian process,  $Y$  is a normal random variable and thus so is  $\sigma Y + \mu\beta$ .

2. We have that

$$X_t \sim \mathcal{N}(\mu t, \sigma^2 t).$$

We denote  $\bar{z}$  the conjugate of  $z$ . The characteristic function is defined as

$$\varphi_{X_t}(\xi) := \mathbb{E}[e^{i\xi X_t}].$$

$$\begin{aligned} \overline{\varphi(\xi)} &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{\mathbb{R}} e^{i\xi x} e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \int_{\mathbb{R}} e^{-i\xi x} e^{-\frac{[x-\mu t]^2}{2\sigma^2 t}} dx \\ &= \frac{1}{\sigma\sqrt{2\pi t}} \mathcal{F} \left( x \mapsto e^{-\frac{(x-\mu t)^2}{2\sigma^2 t}} \right) (\xi). \end{aligned}$$

Where

$$\mathcal{F}(f)(\xi) := \int_{\mathbb{R}} e^{-i\xi x} f(x) dx,$$

denote the Fourier transform of the function  $g$ . By properties of Fourier transform,

$$\mathcal{F}(x \mapsto f(x+h)) = e^{ih\xi} \mathcal{F}(f)(\xi) \quad \text{and} \quad \mathcal{F}(x \mapsto f(ax)) = \frac{1}{a} \mathcal{F}(f)\left(\frac{\xi}{a}\right), \quad a > 0.$$

Therefore,

$$\overline{\varphi_{X_t}(\xi)} = \frac{1}{\sqrt{\pi}} e^{i\mu(t-s)\sigma} \mathcal{F}\left(x \mapsto e^{-x^2}\right)(\sigma\sqrt{2t})$$

The fact that

$$\mathcal{F}\left(x \mapsto e^{-x^2}\right)(\xi) = \sqrt{\pi} e^{-\frac{\xi^2}{4}},$$

is a standard exercise. Combine all these results yields

$$\varphi_{X_t}(\xi) = e^{i\mu t \xi - \frac{\sigma^2 t \xi^2}{2}}.$$

**Remark :** If you knew that  $\mathbb{E}[e^{i\xi B_t}] = e^{-i\frac{t\xi^2}{2}}$ , then I accepted answers as

$$\mathbb{E}[e^{i\xi X_t}] = e^{i\mu t \xi} \mathbb{E}[e^{i\xi \sigma B_t}] = e^{i\xi \mu t - \frac{\sigma^2 t \xi^2}{2}},$$

what make the question quite straightforward.

3. Let  $t > s$ . Remark that  $(X_t)$  has stationary increments, i.e.  $X_t - X_s \sim X_{t-s}$ . Therefore, we immediately have

$$\mathbb{E}[e^{X_t - X_s}] = \varphi_{X_{t-s}}(-i),$$

and thus,

$$\mathbb{E}[e^{X_t - X_s}] = e^{\mu(t-s) + \frac{\sigma^2(t-s)}{2}}.$$

### Exercise 3

Set  $W_t := UB_t$  where  $U$  is s.t.  $UU^T = \mathbf{1}_{m \times m}$  where  $\mathbf{1}_{m \times m}$  denote the identity matrix. The fact that  $W_0 = 0$  and that  $t \mapsto W_t$  is continuous is clear.

The fact that increments are independents is also clear since if two randoms variables  $X$  and  $Y$  are independents, then so are  $f(X)$  and  $f(Y)$  for any measurable function  $f : \mathbb{R}^m \rightarrow \mathbb{R}^m$ . Set  $f(x) = Ux$  where  $x \in \mathbb{R}^m$ . Since  $W_t - W_s = f(B_t - B_s)$  for all  $t > s$  and that increments of  $(B_t)$  are independents, then so are of increments of  $(W_t)$ .

**Claim :** If  $t > s$ , then  $W_t - W_s \sim \mathcal{N}(0, (t-s)\mathbf{1}_{m \times m})$ .

To prove this statement, we just need to show that

$$\mathbb{E}[e^{i\xi^T(W_t - W_s)}] = e^{-\frac{1}{2}(t-s)|\xi|^2}.$$

$$\mathbb{E}[e^{i\xi^T(W_t - W_s)}] = \mathbb{E}[e^{i\xi^T U(B_t - B_s)}] = \mathbb{E}[e^{i(U^T \xi)^T (B_t - B_s)}] = e^{-\frac{1}{2}(t-s)|U^T \xi|^2},$$

where the last equality come from the fact that  $B_t - B_s \sim \mathcal{N}(0, (t-s)\mathbf{1}_{m \times m})$ . Since

$$|U^T \xi|^2 = \langle U^T \xi, U^T \xi \rangle = \xi^T U U^T \xi = \xi^T \xi = |\xi|^2,$$

we finally get

$$\mathbb{E}[e^{i\xi^T(W_t - W_s)}] = e^{-\frac{1}{2}(t-s)|\xi|^2},$$

as wished.

Therefore,  $(W_t)$  is a Brownian motion.

#### Exercise 4

1. Let  $0 \leq t_0 < t_1 < \dots < t_n < \infty$ . Obviously,

$$\mathbb{P}\{Y_{t_1} = 0, \dots, Y_{t_n} = 0\} = 1.$$

Moreover,

$$\mathbb{P}\{X_{t_0} = 0, \dots, X_{t_n} = 0\} = \mathbb{P}([0, \infty) \setminus \{t_0, \dots, t_n\}) = 1,$$

since  $\mathbb{P}\{t_i\} = 0$  for all  $i$  (because there is no atom).

2. Fix  $t \geq 0$ .

$$X_t(\omega) = Y_t(\omega) \iff \omega \in [0, \infty) \setminus \{t\}.$$

Therefore,

$$\mathbb{P}\{X_t = Y_t\} = \mathbb{P}([0, \infty) \setminus \{t\}) = 1.$$

Remark nevertheless that  $\mathbb{P}\{\forall t \geq 0, X_t = Y_t\} = 0$ , and thus they are not indistinguishable.