Probability Theory 2 : Solution Sheet 3

Exercice 2

Suppose (X_t) is a BM.

1. The fact that $X_t \in L^1$ for all t is clear. Moreover, if $t \geq s$

$$\mathbb{E}[X_t \mid \mathcal{F}_s] = \mathbb{E}[X_t - X_s \mid \mathcal{F}_s] + \mathbb{E}[X_s \mid \mathcal{F}_s] \underset{(1)}{=} X_s + \mathbb{E}[X_t - X_s] \underset{(2)}{=} X_s,$$

where (1) come from the fact that X_s is $\mathcal{F}_s$ measurable and $X_t - X_s$ is independent of $\mathcal{F}_s$ and (2) come from the fact that $X_t - X_s \sim \mathcal{N}(0, t-s)$.

2. Since $\mathbb{E}[|X_t^2 - t|] \leq \mathbb{E}[X_t^2] + t = 2t < \infty$, we have $X_t^2 - t \in L^1$ for all t.Moreover, if $t \geq s$,

$$\mathbb{E}[X_t^2 - t \mid \mathcal{F}_s] = \mathbb{E}[(X_t - X_s)^2 + 2X_s(X_t - X_s) + X_s^2 - t \mid \mathcal{F}_s] \underset{(3)}{=} \mathbb{E}[(X_t - X_s)^2] + 2X_s\mathbb{E}[X_t - X_s] + X_s^2 = X_s^2 - s,$$

where we used in (3) the fact that X_s is $\mathcal{F}_s$ measurable and $X_t - X_s$ is independent to $\mathcal{F}_s$.

Exercice 3

1. Set $Z := \sup_{t \geq 0} W_t$.

Claim 1 : If $c > 0$, then $Z \sim cZ$.

Remark that

$$Z > x \iff \exists t \in (0, \infty) \cap \mathbb{Q} : W_t > x.$$

Let (t_k) an enumeration of $(0, \infty) \cap \mathbb{Q}$. Then

$$\mathbb{P}\{Z > x\} = \lim_{n \to \infty} \mathbb{P}\left(\bigcup_{k=1}^{\infty} \{W_{t_k} > x\}\right).$$

Since $(cW_{\frac{t}{c^2}})$ is a Brownian motion, we therefore have

$$\mathbb{P}\{Z > x\} = \mathbb{P}\left\{\sup_{t \geq 0} cW_{\frac{t}{c^2}} > x\right\} = \mathbb{P}\left\{c \sup_{t \geq 0} W_{\frac{t}{c^2}} > x\right\} = \mathbb{P}\{cZ > x\},$$

where the last equality come from the fact that

$$\sup_{t \geq 0} W_t(\omega) = \sup_{t \geq 0} W_{\frac{t}{c^2}}(\omega),$$

for all ω.

Claim 2 : $Z \in \{0, \infty\}$ a.s.

Since

$$\mathbb{P}\{Z > x\} = \mathbb{P}\{cZ > x\} = \mathbb{P}\left\{Z > \frac{x}{c}\right\} \underset{c \to \infty}{\longrightarrow} \mathbb{P}\{Z > 0\},$$

we have

$$\mathbb{P}\{Z > x\} = \mathbb{P}\{Z > 0\}.$$

Therefore

$$\mathbb{P}\{Z = \infty\} = \lim_{n \to \infty} \mathbb{P}\{Z > n\} = \mathbb{P}\{Z > 0\}.$$

Since $W_0 = 0$, we have that $Z \geq 0$ a.s., and thus

$$1 = \mathbb{P}\{Z \geq 0\} = \mathbb{P}\{Z = 0\} + \mathbb{P}\{Z > 0\} = \mathbb{P}\{Z \in \{0, \infty\}\},$$

what prove the claim.

Claim 3 : $\mathbb{P}\{Z=0\}=0$.

Set $Z_t = \sup_{0\leq s\leq t} W_s$. Using reflexion principle,

$$\mathbb{P}\{Z_t>0\}=2\mathbb{P}\{W_t>0\}=1.$$

Since $Z_t\leq Z$ a.s., we get $\mathbb{P}\{Z>0\}=1$ and thus $\mathbb{P}\{Z=0\}=0$.

2. Since $(-W_t)$ is a BM an

$$\inf_{t\geq 0} W_t = -\sup_{t\geq 0}(-W_t),$$

we get

$$\mathbb{P}\{\inf_{t\geq 0} W_t=-\infty\}=\mathbb{P}\{\sup_{t\geq 0}(-W_t)=+\infty\}=1.$$

Exercice 4

1. The argument is to use the fact that the bounded variation of Brownian motion is ∞ a.s. but since differentiability doesn't implies bounded variation as $f(x)=x^2\sin\left(\frac{1}{x^2}\right)$ defined on $[-1,1]$ proved, the statement should have been to prove that the Brownian motion is not continuously differentiable on compact set.

Claim : Continuously differentiable function has bounded variation on $[a,b]$.

Let $f:[a,b]\to\mathbb{R}$ a continuously differentiable function and $\Pi: 0=t_0<t_1<\infty<t_n=b$ a subdivision of $[a,b]$. Then

$$\sum_{i=0}^{n-1}|f(t_{i+1})-f(t_i)|=\sum_{i=0}^{n-1}\left|\int_{t_i}^{t_{i+1}}f'\right|\leq\sum_{i=0}^{n-1}\int_{t_i}^{t_{i+1}}|f'|=\int_a^b|f'|.$$

Therefore,

$$V_a^b(f)\leq\int_a^b|f'|.$$

Since f' is continuous, it's integrable on $[a,b]$, an thus $V_a^b(f)<\infty$, what prove the claim.

2. Recall that

$$\ell:=\limsup_{h\to 0}\frac{|W_{t+h}-W_t|}{h}<k,$$

mean that

$$\forall\varepsilon>0,\exists\delta>0:\forall h\in\mathbb{R},|h|\leq\delta\implies\frac{|W_{t+h}-W_t|}{h}\leq\ell+\varepsilon.$$

So, taking ε small enough to have $\ell+\varepsilon\leq k$, there exist $\delta>0$ s.t.

$$|W_{t+h}-W_t|<k|h|,$$

for all $|h|\leq\delta$. Let $\omega\in A_k$.

(a) If m is fixed, $\left\{[\frac{j}{m},\frac{j+1}{m})\right\}_{j=0}^{m-1}$ is a partition of $[0,1)$, there is $j:=j_m$ s.t $\frac{j}{m}\leq t<\frac{j+1}{m}$. Suppose that m is big enough to have

$$\frac{j}{m}\leq t\leq\frac{j+1}{m}\leq\frac{j+2}{m}\leq\frac{j+3}{m}\leq t+\delta.$$

Then, if $r=1,2,3$,

$$|W_{\frac{j+r}{m}}(\omega)-W_{\frac{j+r+1}{m}}|\leq|W_{\frac{j+r}{m}}(\omega)-W_t(\omega)|+|W_{\frac{j+r+1}{m}}(\omega)-W_t|.$$

Since $|t - \frac{j+r}{m}| \leq \delta$ and $|t - \frac{j+r+1}{m}| \leq \delta$, we have that

$$\begin{aligned} |W_{\frac{j+r}{m}}(\omega) - W_t(\omega)| + |W_{\frac{j+r+1}{m}}(\omega) - W_t| &\leq k\left|t - \frac{j+r}{m}\right| + k\left|t - \frac{j+r+1}{m}\right| \\ &\leq k \cdot \frac{r}{m} + k \cdot \frac{r+1}{m} \\ &= k \cdot \frac{2r+1}{m}, \end{aligned}$$

as wished. The fact that

$$\mathbb{P}\{|W_{t+h} - W_t| \leq a\} \leq \frac{2a}{\sqrt{2\pi h}}, \tag{1}$$

come from the fact that $W_{t+h} - W_t \sim \mathcal{N}(0, h)$.

(b) Let

$$B_{m,j} := \left\{\omega \mid \forall r = 1, 2, 3\ |W_{\frac{j+r}{m}}(\omega) - W_{\frac{j+r+1}{m}}(\omega)| \leq k \cdot \frac{2r+1}{m}\right\},$$

and set

$$B_m = \bigcup_{j=1}^{m} B_{m,j}.$$

Using independence of increments and (1), we have

$$\mathbb{P}(B_{m,j}) = \bigcap_{r=1}^{3} \mathbb{P}\left\{|W_{\frac{j+r}{m}} - W_{\frac{j+r+1}{m}}| \leq k \cdot \frac{2r+1}{m}\right\} \leq \prod_{r=1}^{3} 2 \cdot \frac{2r+1}{\sqrt{2\pi \cdot m}} \sim \frac{1}{m\sqrt{m}}.$$

Therefore

$$\mathbb{P}(B_m) \lesssim \sum_{j=1}^{m} \frac{1}{m\sqrt{m}} = \frac{1}{\sqrt{m}},$$

where $A \lesssim B$ means that there is a constant $C > 0$ s.t. $A \leq CB$. Hence,

$$\mathbb{P}(B_{n_4}) \lesssim \frac{1}{n^2},$$

and thus

$$\sum_{n=1}^{\infty} \mathbb{P}(B_{n^4}) < \infty.$$

The claim follow by Borel-Cantelli lemma.

(c) Let $\omega \in D_0$. Let t s.t.

$$\lim_{h \to 0} \frac{W_{t+h} - W_t}{h},$$

exists. In particular, if

$$\ell := \limsup_{n \to \infty} \frac{|W_{t+h}(\omega) - W_t(\omega)|}{h},$$

and $k = \lfloor \ell \rfloor + 2$, then $\omega \in A_k$. Therefore,

$$D_0 \subset \bigcup_{k \in \mathbb{N}} A_k =: G_0.$$

Since

$$\bigcup_{k \in \mathbb{N}} A_k \subset \limsup_{n \to \infty} A_{n^2},$$

we get $\mathbb{P}(G_0) = 0$ as wished.

(d) Set

$$D_n = \{\omega \mid W_t(\omega) \text{ is differentiable for a least one } t \in [n, n+1)\},$$

and

$$A_k^n = \left\{\omega \mid \limsup_{h\to 0} \frac{|W_{t+h} - W_t|}{h} < k \text{ for at least on } t \in [n, n+1)\right\}.$$

We have that D_n and $\bigcup_{k\in\mathbb{N}} A_k^n =: G_n$ are nulle sets, and thus, $\bigcup_{n\in\mathbb{N}} D_n$ is a null set, what prove the claim.