Probability Theory 2 : Solution Sheet 14

Exercice 1

Apply Itô formula to $g(t,x) := f(t)x$. Notice that even if g is not $\mathcal{C}^2$, what matter is that $x \mapsto g(t,x)$ is $\mathcal{C}^2$ for all t and $t \mapsto g(t,x)$ is $\mathcal{C}^1$ for all x, which is obviously the case.

Exercice 2

1. Left to the reader.
2. We want to find f s.t.

$$\begin{cases} \partial_x f(t,x) = H_n(t,x) \\ \partial_t f(t,x) + \frac{1}{2}\partial_{xx}^2 f(t,x) = 0 \\ f(t,x) = H_{n+1}(t,x) \\ f(0,0) = 0 \end{cases}.$$

Using the previous question, we see that $f(t,x) := H_{n+1}(x,t)$ solve the previous system. Applying Itô to f yield the wished result. Since H_n is continuous on $(0,\infty)\times\mathbb{R}$, it's in $M^2[0,T]$ for all $T>0$ (be sure to make the details properly is there is such question at the exam). Therefore, $(H_{n+1}(t,B_t))_{[0,T]}$ is a martingale for all $T>0$, and thus, $(H_n(t,B_t))_{t\geq 0}$ is a Martingale.

Exercice 3

The result follow from the following theorem :

Théorème 0.1.

Let $X, Y \in M^2[a,b]$ and $A \in \mathcal{F}$ s.t. $X_t = Y_t$ a.s. for all $t \in [a,b]$. Then

$$\int_a^b X_t \,\mathrm{d}B_t = \int_a^b Y_t \,\mathrm{d}B_t.$$

Démonstration. The proof can be find in the book Stochastic Calculs of P. Baldi. Let $\pi_n : a = t_0^n < \ldots < t_n^n = b$ s.t. $t_i = a + \frac{i}{n}$, $i = 0,\ldots,n$. For $Z \in M^2[a,b]$ denote

$$\mathcal{G}_n Z(t) = \sum_{i=0}^{n-1} Z^i \mathbf{1}_{[t_i^n, t_{i+1}^n)}(t),$$

where

$$Z_i := \frac{1}{t_i^n - t_{i-1}^n}\int_{t_{i-1}^n}^{t_i^n} Z_t \,\mathrm{d}t \in \mathcal{F}_{t_i^n}.$$

One can prove that $\mathcal{G}_n Z \in L^2(a,b)$ and that $\mathcal{G}_n Z \to Z$ in $L^2(a,b)$ when $n\to\infty$. Since $X_t = Y_t$ a.s. on A for all $t\in[a,b]$, we have that

$$P_n(t) := \mathcal{G}_n X = \mathcal{G}_n Y =: Q_n(t) \quad \text{a.s. on } A,$$

for all $t \in [a,b]$. Therefore,

$$\int_a^b P_n(t)\,\mathrm{d}B_t = \int_a^b Q_n(t)\,\mathrm{d}B_t, \quad \text{a.s. on } A.$$

Since $P_n \to X$ and $Q_n \to Y$ in $L^2(a,b)$ when $n\to\infty$, we finally get

$$\int_a^b P_n(t)\,\mathrm{d}B_t \xrightarrow[n\to\infty]{L^2} \int_a^b X_t\,\mathrm{d}B_t,$$

$$\int_a^b Q_n(t)\,\mathrm{d}B_t \xrightarrow[n\to\infty]{L^2} \int_a^b Y_t\,\mathrm{d}B_t,$$

and thus wished result, since if $Z_n \to Z$ in L^2, there is a subsequence that convergence to Z a.s. $\square$