Probability Theory 2 : Solution Sheet 11

Exercice 1

Let (S_t) a supermartingale. The implication is obvious since martingale has constant expectation. Conversely, suppose that (S_t) has constant expectation. Let $s < t$. Since (S_t) is a supermartingale,

$$S_s - \mathbb{E}\left[S_t \mid \mathcal{F}_s\right] \geq 0.$$

Moreover, since (S_t) has constant expectation

$$\mathbb{E}\big[S_s - \mathbb{E}\left[S_t \mid \mathcal{F}_s\right]\big] = 0.$$

Therefore,

$$\mathbb{E}[S_t \mid \mathcal{F}_s] = S_s.$$

Exercice 2

We have that

$$M_t = \int_0^t V_s \, \mathrm{d}B_s = \sum_{k=0}^{\ell_t - 1} \eta_k (B_{t_{k+1}} - B_{t_k}) + \eta_{\ell_t}(B_t - B_{\ell_t}),$$

where $\ell_t = \lfloor t \rfloor$. We can suppose WLOG that $t \in \mathbb{N}$ (do the proof whenever $t \notin \mathbb{N}$ to convince yourself). To simplify notation, we denote

$$\Delta B_i := B_{t_{i+1}} - B_{t_i} \quad \text{and} \quad \Delta t_i = t_{i+1} - t_i.$$

Let $s < t$. We suppose WLOG that $s \in \mathbb{N}$ (do the proof whenever $s \notin \mathbb{N}$ to convince yourself). Then, using classical properties of Brownian motion and conditional expectation (technical details are left to the reader)

$$\begin{aligned}
&\mathbb{E}[M_t^2 - \langle M \rangle_t \mid \mathcal{F}_s] = \mathbb{E}\left[\sum_{k=0}^{t-1}\sum_{i=0}^{t-1} \eta_i \eta_k \Delta B_i \Delta B_k - \sum_{k=0}^{t-1} \eta_k^2 \Delta t_k \mid \mathcal{F}_s\right] \\
&= \underbrace{\sum_{k=0}^{s-1}\sum_{i=0}^{s-1} \eta_k \eta_i \Delta B_i \Delta B_k - \sum_{k=0}^{s-1} \eta_k^2 \Delta t_k}_{=M_s^2 - \langle M \rangle_s} + \sum_{k=0}^{s-1}\sum_{i=s}^{t-1} \underbrace{\mathbb{E}[\eta_k \eta_i \Delta B_i \Delta B_k \mid \mathcal{F}_s]}_{=\mathbb{E}[\eta_i \eta_k \Delta B_k | \mathcal{F}_s] \cdot \mathbb{E}[\Delta B_i] = 0} \\
&+ \sum_{k=s}^{t-1}\sum_{i=0}^{s-1} \underbrace{\mathbb{E}[\eta_k \eta_i \Delta B_i \Delta B_k \mid \mathcal{F}_s]}_{=\mathbb{E}[\eta_i \eta_k \Delta B_i | \mathcal{F}_s] \cdot \mathbb{E}[\Delta B_k] = 0} + \underbrace{\sum_{k=s}^{s-1}\sum_{i=s}^{t-1} \mathbb{E}[\eta_k \eta_i \Delta B_i \Delta B_k \mid \mathcal{F}_s] - \sum_{k=s}^{t-1} \mathbb{E}[\eta_k^2 \mid \mathcal{F}_s] \Delta t_k}_{=:I} \\
&= M_s^2 - \langle M \rangle_s ,
\end{aligned}$$

where I has been computed as follow :

$$I = \sum_{k=s}^{t-1} \underbrace{\mathbb{E}[\eta_k^2 (\Delta B_k)^2 \mid \mathcal{F}_s]}_{=\mathbb{E}[\eta_k^2 | \mathcal{F}_s] \Delta t_k} - \sum_{k=s}^{t-1} \mathbb{E}[\eta_k^2 \mid \mathcal{F}_s] \Delta t_k + \sum_{k=s}^{t-1} \sum_{\substack{i=s \\ i \neq k}}^{t-s} \underbrace{\mathbb{E}[\eta_k \eta_i \Delta B_k \Delta B_i \mid \mathcal{F}_s]}_{=0} = 0.$$

Exercice 3

1. Let (τ_n) a regularizing sequence. Since $|M_{t \wedge \tau_n}| \leq \sup_{0 \leq s \leq t} |M_s| \in L^1(\Omega)$, the claim follow by DCT.

2. Denote $\boldsymbol{B} = (B^1, B^2, B^3)$.

(a) Let $t \geq 1$. Set $h(x,y,z) = \frac{1}{\sqrt{x^2+y^2+z^2}} = \frac{1}{\|x\|}$. We have $\nabla h(x) = \frac{x}{\|x\|^3}$. Since h is $\mathcal{C}^2(\mathbb{R} \setminus \{0\})$ and $B_s \neq 0$ for all $s \in [0,t]$ a.s. we can use Itô formula. By Itô formula,

$$\begin{aligned}
&h(B_1(t), B_2(t), B_3(t)) \\
&= h(B_1^1, B_1^2, B_1^3) + \int_1^t \nabla h(B_s^1, B_s^2, B_s^3) \cdot \mathrm{d}\boldsymbol{B}_s + \int_1^t \frac{1}{2} \Delta h(B_s^1, B_s^2, B_s^3)\,\mathrm{d}s \\
&= h(B_1^1, B_1^2, B_1^3) + \int_1^t \nabla h(B_s^1, B_s^2, B_s^3) \cdot \mathrm{d}\boldsymbol{B}_s.
\end{aligned}$$

Denote

$$h_i(x) := h_i(x_1, x_2, x_3) = \frac{x_i}{\|x\|^3}.$$

Let

$$\tilde{\Omega} := \{\omega \mid \forall t > 0, B_t(\omega) \neq 0\} \cap \{\omega \mid t \mapsto B_t(\omega) \text{ continuous}\}.$$

If $\omega \in \tilde{\Omega}$, by then there is $C = C(\omega) > 0$ s.t. $\|B_t(\omega)\| \geq C(\omega) > 0$. Therefore, $s \mapsto h(B_s(\omega))$ is continuous on $[0,t]$, and thus $s \mapsto h_i(B_s(\omega))$ is in $L^2([1,t])$. Since $\mathbb{P}(\tilde{\Omega}) = 1$, we get $h(B_t) \in M^2_{loc}([1,t])$. Therefore,

$$\int_0^t h_i(B_s)\,\mathrm{d}B_s^i,$$

is a local martingale for all $i = 1,2,3$. Since a finite sum of local martingale is a local martingale, we conclude that $(h(B_t))_{t \geq 1}$ is a local martingale.

(b) Using polar coordinates yields

$$\mathbb{E}[M_t^2] = \int_{\mathbb{R}^3} \frac{1}{x^2+y^2+z^2} e^{-\frac{x^2+y^2+z^2}{2t}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z = \frac{1}{t}.$$

(c) If (M_t) would be a Martingale, (M_t^2) would be a submartingale, an thus, $\mathbb{E}[M_t^2]$ would be increasing. Since $t \mapsto \frac{1}{t}$ is strictly decreasing, (M_t) is not a martingale.

Exercice 4

1. (a) Set $f(x,t) = e^{\frac{t}{2}} \sin(x)$. By Itô formula

$$\begin{aligned}
X_t = e^{\frac{t}{2}} \sin(B_t) &= f(B_t, t) \\
&= \int_0^t \frac{\partial f}{\partial x}(s, B_s)\,\mathrm{d}B_s + \frac{1}{2}\int_0^t \frac{\partial^2 f}{\partial x^2}(s, B_s)\,\mathrm{d}s + \int_0^t \frac{\partial f}{\partial s}(s, B_s)\,\mathrm{d}s \\
&= \int_0^t e^{\frac{s}{2}} \cos(B_s)\,\mathrm{d}s.
\end{aligned}$$

Since

$$\mathbb{E}\left[\int_0^t e^s \sin^2(B_s)\,\mathrm{d}s\right] \leq e^t - 1 < \infty,$$

X is a martingale.

(b) Using the same method,

$$Y_t = 1 - \int_0^t e^{\frac{s}{2}} \sin(B_s)\,\mathrm{d}B_s,$$

which is also a martingale.

(c) By properties of Itô integral, $\mathbb{E}[X_t] = 0$ and $\mathbb{E}[Y_t] = 1$ for all t.

2. There are obviously Itô processes. Moreover,

$$\langle X, Y \rangle_t = \int_0^t e^s \cos(B_s) \sin(B_s)\,\mathrm{d}s.$$