Probability Theory 2 : Solution Sheet 10

Exercice 1

1. Since

$$\int_0^T \mathbf{1}_{\{t\leq\tau_n\}} X_t^2 \,\mathrm{d}t = \int_0^{\tau_n} X_t^2 \,\mathrm{d}t \leq n,$$

then $(X_t \mathbf{1}_{\{t\leq\tau_n\}})$ is in $M^2([0,T])$. Therefore, using Itô isometry yield,

$$\begin{aligned}
\mathbb{E}\left[\int_0^{\tau_n} X_t^2 \,\mathrm{d}t\right] &= \mathbb{E}\left[\int_0^T \mathbf{1}_{\{t<\tau_n\}} X_t^2 \,\mathrm{d}t\right] \\
&= \mathbb{E}\left[\left(\int_0^T \mathbf{1}_{\{s<\tau_n\}} X_s \,\mathrm{d}B_t\right)^2\right] \\
&= \mathbb{E}\left[\left(\int_0^{\tau_n} X_t \,\mathrm{d}B_t\right)^2\right] \\
&= \mathbb{E}[M_{t\wedge\tau_n}^2] \\
&\leq \mathbb{E}\left[\sup_{0\leq t\leq T} M_t^2\right].
\end{aligned}$$

2. If $X \in M^2([0,T])$, then $(M_t)_{t\in[0,T]}$ is a square integrable martingale, and thus, by Doob inequality,

$$\mathbb{E}\left[\sup_{0\leq t\leq T} M_t^2\right] \leq 4\mathbb{E}\left[M_T^2\right] = 4\mathbb{E}\left[\int_0^T X_t^2 \,\mathrm{d}t\right],$$

where we used Itô isometry in the last equality.

Conversely, suppose $\mathbb{E}\left[\sup_{0\leq t\leq T} M_t^2\right] < \infty$. Let $\tau_n \nearrow \infty$ a.s. By monotone convergence theorem and the previous question,

$$\mathbb{E}\left[\int_0^T X_s^2 \,\mathrm{d}s\right] = \lim_{n\to\infty} \mathbb{E}\left[\int_0^{\tau_n} X_s^2 \,\mathrm{d}s\right] \leq \mathbb{E}\left[\sup_{0\leq t\leq T} M_t^2\right] < \infty.$$

Therefore, $(X_t) \in M^2([0,T])$.

Exercice 2

This exercise is only calculation. We just sketch the solution. Since $Z \sim \mathcal{N}(0,1)$, its density function is given by $f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}$.

1.

$$\mathbb{E}[Z^2 e^{\alpha Z}] = \int_{\mathbb{R}} z^2 e^{\alpha z} f_Z(z) \,\mathrm{d}z.$$

2. (H_t) is continuous on $\{B_1 = 0\}^c$. Since $\mathbb{P}\{B_1 = 0\} = 0$, (H_t) is a.s. continuous and $\lim_{t\to 1^-} H_t = 0$ a.s. The fact that $(H_t) \in M_{loc}^2([0,1])$ is clear since $t \mapsto H_t$ is a.s. continuous on $[0,1]$. To check that $(H_t) \notin M^2([0,1])$, we can prove that

$$\mathbb{E}\left[\int_0^1 H_s^2 \,\mathrm{d}s\right] \underset{(1)}{=} \int_0^1 \mathbb{E}\left[H_s^2\right] \,\mathrm{d}s = +\infty,$$

where (1) follow by Fubini.

Exercice 3

1. Let (τ_n) and (σ_n) sequences of stopping time s.t. $\tau_n, \sigma_n \nearrow \infty$ a.s. and s.t. $(M_{t\wedge\tau_n})$ and $(N_{t\wedge\sigma_n})$ (such sequences are called *regularizing sequences*. Are continuous martingales for all n. In particular, $(M_{(t\wedge\tau_n)\wedge\sigma_n})$ and $(N_{(t\wedge\sigma_n)\wedge\tau_n})$ are martingale, and thus, $\kappa_n := \sigma_n \wedge \tau_n$ is a sequence of stopping s.t. $\kappa_n \nearrow \infty$ and s.t. $\big((M+N)_{t\wedge\kappa_n}\big)$ is a Martingale for all n. Therefore, $M+N$ is a local martingale.
2. Take $M = N = B$, where B is a Brownian motion. M and N are local Martingales (because there are martingales), but $NM = B^2$ is not.
3. Suppose that M is a positive local martingale. To prove that M is a supermartingale, it's enough to prove that if $s < t$, then for all $A \in \mathcal{F}_s$,

$$\mathbb{E}[M_t \mathbf{1}_A] \leq \mathbb{E}[M_s \mathbf{1}_A]. \tag{1}$$

Indeed, if (1) hold, then

$$\mathbb{E}\big[\mathbb{E}[M_t \mid \mathcal{F}_s]\mathbf{1}_A\big] \leq \mathbb{E}[M_s \mathbf{1}_A],$$

for all $A \in \mathcal{F}_s$, and since $\mathbb{E}[M_t \mid \mathcal{F}_s]$ and M_s are $\mathcal{F}_s$ measurable, we'll get $\mathbb{E}[M_t \mid \mathcal{F}_s] \leq M_s$ a.s.

Let (τ_n) a regularizing sequence and let $A \in \mathcal{F}_s$. Then

$$\begin{aligned}
\mathbb{E}[\mathbf{1}_{A\cap\{\tau_n > s\}}(M_{t\wedge\tau_n} - M_s)] &= \mathbb{E}\big[\mathbf{1}_{A\cap\{\tau_n>s\}}(M_{t\wedge\tau_n} - M_{s\wedge\tau_n})\big] \\
&= \mathbb{E}\Big[\mathbf{1}_{A\cap\{\tau_n>s\}}\Big(M_{t\wedge\tau_n} - \mathbb{E}\left[M_{t\wedge\tau_n} \mid \mathcal{F}_s\right]\Big)\Big] \\
&= 0,
\end{aligned}$$

where the last equality come from the fact that $A \cap \{\tau_n > s\}$ is $\mathcal{F}_s$ measurable and that

$$\mathbb{E}\big[\mathbb{E}[\mathbf{1}_{A\cap\tau_n>s} M_{t\wedge\tau_n} \mid \mathcal{F}_s]\big] = \mathbb{E}[M_{t\wedge\tau_N}\mathbf{1}_{A\cap\{\tau_n>s\}}].$$

Remark that

$$\lim_{n\to\infty} \mathbf{1}_{A\cap\{\tau_n>s\}}(M_{t\wedge\tau_n} - M_s) = \mathbf{1}_A(M_t - M_s),$$

and

$$\mathbf{1}_{A\cap\{\tau_n>s\}}(M_{t\wedge\tau_n} - M_s) \geq -M_s.$$

Moreover, $M_s \in L^1(\Omega)$ for all $s \geq 0$. Indeed, by Fatou's lemma,

$$\mathbb{E}[M_s] \leq \liminf_{n\to\infty} \mathbb{E}[M_{s\wedge\tau_n}] = \mathbb{E}[M_0] < \infty.$$

Therefore, using Fatou's lemma yields

$$0 = \liminf_{n\to\infty} \mathbb{E}[\mathbf{1}_{A\cap\{\tau_n>s\}}(M_{t\wedge\tau_n} - M_s)] \geq \mathbb{E}[\mathbf{1}_A(M_t - M_s)],$$

and thus, (1) follow.