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Probability : Practice Exam

31st January from 13 :45 to 16 :00

The total amount of points is 81. A mark is sufficient if 30 points are obtained (25 points is nevertheless acceptable). The mark is maximal if you have 60 or more. Don't spend to much time an a question. There are many questions that have been made in the exercises sheets or during the lecture : try to do them first.

Good Luck !

Note : In all this practice exam, if $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space, we say that X is a *real random variable* if $X : \Omega \longrightarrow \mathbb{R}$ is a random variable (in particular $X(\omega) \in \mathbb{R}$ for all $\omega \in \Omega$).

Problem 1 (15 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

1. (a) Define rigorously each element of the triplet. [3 pts]

(b) Give the definition of a real random variable on this probability space. [1 pt]

2. Let $\{A_i\}_{i\in\mathbb{N}}$ a collection of $\mathcal{F}$. Show that $\displaystyle\bigcap_{n=0}^{\infty} A_n \in \mathcal{F}$. [2 pts]

3. (a) Let $B \in \mathcal{F}$. We denote [5 pts]

$$\mathcal{F}_B := \{A \cap B \mid A \in \mathcal{F}\}.$$

Let define $\mathbb{Q} : \mathcal{F}_B \longrightarrow [0, 1]$ by

$$\mathbb{Q}(K) = \frac{\mathbb{P}(K)}{\mathbb{P}(B)}, \quad K \in \mathcal{F}_B.$$

Prove that $(B, \mathcal{F}_B, \mathbb{Q})$ is a probability space.

Remark : If $A \in \mathcal{F}_B$, then A^c denote the complementary of A in B (and not in Ω). It means that $A^c = B \setminus A$. However, if $C \in \mathcal{F}$, then C^c is the complementary in Ω, i.e. $C^c = \Omega \setminus C$.

(b) Let $\Omega = \big\{(P, P), (P, F), (F, P), (F, F)\big\}$ and

$$\mathcal{F} = \sigma\Big\{\big\{(P, F), (F, P)\big\}, \big\{(P, P)\big\}, \big\{(F, F)\big\}\Big\}.$$

Let $X : \Omega \longrightarrow \mathbb{R}$ defined by

$$X = \mathbf{1}_{\{(P,F)\}}$$

(i) Is X a random variable on $(\Omega, \mathcal{F})$? *(Justify !)* [2 pts]

(ii) Is $X|_B$ a random variable on $(B, \mathcal{F}_B, \mathbb{Q})$? *(Justify !)* [2 pts]

Indication : The function $X|_B$ is the restriction of X on B, i.e. $X|_B : B \longrightarrow \mathbb{R}$ is defined by $X|_B(\omega) = X(\omega)$ for all $\omega \in B$.

Problem 2 (6 points)

A restaurant with 150 seats noticed that 20% of the people who reserved a table finally don't come. The fact that someone who made a reservation and decides not to come is independent of all other clients. We suppose that each person can only make a reservation for himself. It means that a person can't reserve a table for several people (so, in this restaurant you can't reserve a table for you and your friend). We also suppose that to eat in this restaurant, you have to make a reservation.

1. If the restaurant takes 190 reservations, what is the probability that every one who decides to come has a seat ? [3 pts]

2. How many reservation can take the restaurant so that each person has a place with probability 0.99 ? [3 pts]

Hint : $\Phi_{0,1}(2.33) = 0.99$. *Just give the right equation, but it's not necessary to solve it.*

Problem 3 (15 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a discrete probability space (i.e. Ω is countable). Let X and Y two reals randoms variables.

1. (a) Give the definition of the expectation and of the variance of X. [2 pts]

(b) Give the definition of the covariance of X and Y. [1 pt]

(c) Give the definition of the independence of X and Y. [1 pt]

2. Prove the following statements :

(a) $\operatorname{Var}(aX + bY) = a^2\operatorname{Var}(X) + b^2\operatorname{Var}(Y) + 2ab\operatorname{Cov}(X, Y)$. [3 pts]

(b) If X and Y are independent, then $\operatorname{Cov}(X, Y) = 0$. [2 pts]

3. Consider the space $\Omega = \{1, \ldots, 6\}$, the probability measure $\mathbb{P}\{\omega\} = \frac{1}{6}$ and the randoms variables

$$X = \mathbf{1}_{\{3,4\}} + 2 \cdot \mathbf{1}_{\{5,6\}} \quad \text{and} \quad Y = \mathbf{1}_{\{1,3,5\}}.$$

(a) Are these randoms variables independents ? *(Justify !)* [3 pts]

(b) Compute $\operatorname{Var}\big(e^3 X - \sqrt{5} Y\big)$. [3 pts]

Problem 4 (6 points)

Peter forgets his bag in a shop with probability $\frac{1}{4}$. Today, he took his bag to go shopping, and was in four different shops. Given that he went back home without his bag, what is the probability that peter forgot his bag in the last shop ?

Hint : Binomial works, but it's not the best way to solve this problem.

Problem 5 (8 points)

Consider two different dices : a dice A that contain 4 red faces and 2 black faces and a dice B that contain 2 red faces and 4 black faces. The game goes as follow : we toss a fair coin. If the coin shows « *Head* », we take the dice A and always use this dice for all the game. If the coin shows « *Tail* », then we use the dice B for the game. We denote the events

$$R_i : \text{« The dice shows red at } i^{th} \text{ thrown »}$$

$$B_i : \text{« The dice shows black at } i^{th} \text{ thrown »}.$$

1. Compute the probability that we have a red face appear at the first thrown. [2 pts]

2. Given that at the first thrown we had a red face, what is the probability to used the dice A ? [2 pts]

3. Given that at the first thrown we had a red face, what is the probability to have a have a red face at the second thrown ? [2 pts]

4. Are the result of the first thrown and of the thrown independents ? *(Justify !)* [2 pts]

Turn please $\longrightarrow$

Problem 6 (11 points)

Let consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. We denote $\mathcal{B}(\mathbb{R})$ the Borel $\sigma-$algebra of $\mathbb{R}$.

1. Suppose that X is a real random variable. Show that $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ is a probability space where $\mathbb{P}_X$ is defined by [3 pts]

$$\mathbb{P}_X(A) = \mathbb{P}\{X \in A\},$$

for all Borel set A. We denote $\mathbb{E}_X$ the expectation with respect to the measure $\mathbb{P}_X$.

Suppose that X is a real random variable that follow an $Exp(1)$ distribution. It mean that

$$\mathbb{P}\{X \in A\} = \int_A e^{-x} \mathbf{1}_{[0,\infty)}(x)\, \mathrm{d}x.$$

2. Let Y and Z reals random variables on the probability space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), \mathbb{P}_X)$ defined by

$$Y = \mathbf{1}_{[0,2]} + \mathbf{1}_{[1,3]} \quad \text{and} \quad Z = \mathbf{1}_{[1,5]}.$$

(a) Compute $\mathbb{E}_X[Y]$, $\mathbb{E}_X[Z]$ and $\mathbb{E}_X[YZ]$. [3 pts]

(b) Are Y and Z independents? *(Justify!)* [1 pt]

Just give the idea of the calculations and the right argument, but you don't have to do the calculation rigorously.

3. (a) Show that $Z \circ X$ is a random variable on $(\Omega, \mathcal{F}, \mathbb{P})$ where $(Z \circ X)(\omega) = Z(X(\omega))$ for all $\omega \in \Omega$. [2 pts]

(b) Compute $\mathbb{E}[Z \circ X]$ where $\mathbb{E}$ is the expectation with respect to the measure $\mathbb{P}$. [2 pts]

Problem 7 (12 points)

Let $(\Omega, \mathcal{F}, \mathbb{P})$ a probability space.

1. Let $A, B \in \mathcal{F}$. Prove the following assertions :

(a) i. If $B \subset A$, prove that [2 pts]

$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(B).$$

ii. Is this formula still true if we don't suppose $B \subset A$? *(If yes, prove it; if not, provide a counter-example)* [2 pts]

(b) Suppose A and B are independents. Prove that A and B^c are independents as well. [2 pts]

2. Let $\{B_i\}_{i \in \mathbb{N}} \subset \mathcal{F}$ a collection of decreasing elements, i.e. $B_{n+1} \subset B_n$ for all $n \in \mathbb{N}$. Prove that [3 pts]

$$\mathbb{P}\left(\bigcap_{n \in \mathbb{N}} B_n\right) = \lim_{n \to \infty} \mathbb{P}(B_n).$$

3. Let $(X_n)_{n \in \mathbb{N}}$ a sequence of real random variables. We suppose that $(X_n)_{n \in \mathbb{N}}$ is almost surely increasing, i.e. $\mathbb{P}\{X_n \leq X_{n+1}\} = 1$ for all n. Show that for all $x \in \mathbb{R}$, [3 pts]

$$\mathbb{P}\left(\bigcap_{n \in \mathbb{N}} \{X_n \leq x\}\right) = \lim_{n \to \infty} \mathbb{P}\{X_n \leq x\}.$$

Problem 8 (8 points)

In Germany, there were $40,000$ mariages during the year 2018.

1. (a) Provide a probability space to describe the number of couples that are both born on the 24th of January. [2 pts]

(b) Give an approximation of the probability that exactly 25 couples are born on the 24th of January. [2 pts]

2. (a) Provide a probability space to describe the number of couple that are both born on the same day. [2 pts]

(b) Give an approximation of the probability that exactly 25 couples are born on the same day. [2 pts]

- THE END -

Name :

Ex 1	**Ex 2**	**Ex 3**	**Ex 4**	**Ex 5**	**Ex 6**	**Ex 7**	**Ex 8**
15 pts	6 pts	15 pts	6 pts	8 pts	11 pts	12 pts	8 pts

<u>**Total :**</u> **/60**