Prof. Dr. M. Hinz
Christo Schreier
Bielefeld University

Probability : Practice Exam (solution)

Problem 1

1. See your notes.

2. Let $\{A_i\}_{i=1}^{\infty}$ a collection of $\mathcal{F}$. Since $\mathcal{F}$ is a σ a $\sigma-$algebra, it's stable by complementary and countable union. Therefore,

$$\bigcap_{i=1}^{\infty} A_i = \left(\bigcup_{i=1}^{\infty} A_i^c\right)^c \in \mathcal{F}.$$

3. (a) Let prove that $\mathcal{F}_B$ is a $\sigma-$algebra. The fact that B and $\varnothing$ are in $\mathcal{F}_B$ is clear. Let $A \in \mathcal{F}_B$. i.e. there is $C \in \mathcal{F}$ s.t. $A = C \cap B$. Then,

$$A^c = B \setminus A = B \setminus (C \cap B) = (B \cap C^c) \cup (B \cap B^c) = B \cap C^c.$$

Here $C^c = \Omega \setminus C \in \mathcal{F}$. Therefore, $A^c \in \mathcal{F}_B$. Let $\{A_i\}_{i=1}^{\infty}$ a collection of element of $\mathcal{F}_B$. I.e. for all A_i there is $C_i \in \mathcal{F}$ s.t. $A_i = B \cap C_i$. Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (C_i \cap B) = B \cap \underbrace{\left(\bigcup_{i=1}^{\infty} C_i\right)}_{\in \mathcal{F}} \in \mathcal{F}_B.$$

Let show that $\mathbb{Q}$ is a probability on $(B, \mathcal{F}_B)$. We have $\mathbb{Q}(B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$. Also, if $\{A_i\}_{i=1}^{\infty}$ is disjoint collection of $\mathcal{F}_B$, then

$$\mathbb{Q}\left(\bigcup_{i=1}^{\infty} A_i\right) = \frac{1}{\mathbb{P}(B)} \mathbb{P}\left(\bigcup_{i=1}^{\infty} A_i\right) \underset{(*)}{=} \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \mathbb{Q}(A_i),$$

where $(*)$ come from the fact that $\mathbb{P}$ is a probability on $(\Omega, \mathbb{F})$ and $A_i \in \mathcal{F}$ for all i.

(b) i. It's not a random variable because $X^{-1}(\{1\}) = \{(P, F)\} \notin \mathcal{F}$.

ii. It is a random variable because $X|_B = 0$ and thus

$$X^{-1}(B) = \begin{cases} \varnothing & 0 \notin B \\ B & 0 \in B. \end{cases}$$

Therefore $X^{-1}(B) \in \mathcal{F}_B$ for all Borel set B.

Problem 2

Let X_i denote « *The i^{th} client comes* ». Then $X_i \sim Bern(0.8)$. Set

$$S_n = X_1 + \ldots + X_n.$$

Since the X_i are i.i.d., we have that $S_n \sim Binom(n, 0.8)$.

1. Using the DeMoivre Laplace theorem

$$\mathbb{P}\{S_{190} \le 150\} = \mathbb{P}\left\{\frac{S_{190} - 190 \cdot 0.8}{\sqrt{190 \cdot 0.2 \cdot 0.8}} \le \frac{150 - 190 \cdot 0.8}{\sqrt{190 \cdot 0.2 \cdot 0.8}}\right\} = \Phi_{0,1}(-0.36) \approx 0.36.$$

2. We have to find n s.t. $\mathbb{P}\{S_n \le 150\} \ge 0.99$. This equation is equivalent to

$$\mathbb{P}\left\{\frac{S_n - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}} \le \frac{150 - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}}\right\} = \Phi_{0,1}(2.33) \iff \frac{150 - 0.8n}{\sqrt{n \cdot 0.2 \cdot 0.8}} \ge 2.33 \iff n \le 172.$$

So, the restaurant can take at most 172 reservation.

Problem 3

1. See your notes.

2. (a)

$$\begin{aligned}\mathrm{Var}(aX+bY) &= \mathbb{E}\Big[\big(aX+bY-\mathbb{E}[aX+bY]\big)^2\Big]\\ &= \mathbb{E}\Big[\big((aX-\mathbb{E}[aX])+(bY-\mathbb{E}[bY])\big)^2\Big]\\ &= \mathbb{E}\Big[\big(aX-\mathbb{E}[aX]\big)^2\Big]+2\mathbb{E}\Big[\big(aX-\mathbb{E}[aX]\big)\big(bY-\mathbb{E}[bY]\big)\Big]+\mathbb{E}\Big[\big(bY-\mathbb{E}[bY]\big)^2\Big]\\ &= a^2\mathbb{E}\Big[\big(X-\mathbb{E}[X]\big)^2\Big]+2ab\mathbb{E}\Big[\big(X-\mathbb{E}[X]\big)\big(Y-\mathbb{E}[Y]\big)\Big]+b^2\mathbb{E}\Big[\big(Y-\mathbb{E}[Y]\big)^2\Big]\\ &= a^2\mathrm{Var}(X)+b^2\mathrm{Var}(Y)+2ab\mathrm{Cov}(X,Y).\end{aligned}$$

(b) If X and Y are independent, we know that $\mathbb{E}[XY]=\mathbb{E}[X]\mathbb{E}[Y]$, and thus

$$\begin{aligned}\mathrm{Cov}(X,Y) &= \mathbb{E}\Big[\big(X-\mathbb{E}[X]\big)\big(Y-\mathbb{E}[Y]\big)\Big]\\ &= \mathbb{E}\Big[XY-X\mathbb{E}[Y]-Y\mathbb{E}[X]+\mathbb{E}[X]\mathbb{E}[Y]\Big]\\ &= \mathbb{E}[XY]-\mathbb{E}[X]\mathbb{E}[Y]\\ &= 0.\end{aligned}$$

3. (a) We have

$$\begin{aligned}\mathbb{P}\{X=0,Y=0\}=\mathbb{P}\{1\}=\frac{1}{6}=\mathbb{P}\{X=0\}\mathbb{P}\{Y=0\},\\ \mathbb{P}\{X=1,Y=0\}=\mathbb{P}\{3\}=\frac{1}{6}=\mathbb{P}\{X=1\}\mathbb{P}\{Y=0\}\\ \mathbb{P}\{X=2,Y=0\}=\mathbb{P}\{5\}=\frac{1}{6}=\mathbb{P}\{X=2\}\mathbb{P}\{Y=0\}.\end{aligned}$$

Notice that since A and B independent implies that A and B^c are independent (proved in problem 6 of the sheet 2 and also is problem 6 of this exam), we can conclude that X and Y are independent.

(b) Using **2.** and the fact that X and Y are independent, we get

$$\mathrm{Var}(e^3X-\sqrt{5}Y)=e^6\mathrm{Var}(X)+5\mathrm{Var}(X).$$

Now,

$$\begin{aligned}\mathbb{E}[X]=\mathbb{P}\{3,4\}+2\mathbb{P}\{5,6\}=1,\\ \mathbb{E}[X^2]=\mathbb{P}\{3,4\}+4\mathbb{P}\{5,6\}=\frac{5}{3},\end{aligned}$$

and thus

$$\mathrm{Var}(X)=\mathbb{E}[X^2]-\mathbb{E}[X]^2=\frac{2}{3}.$$

Also

$$\mathbb{E}[Y]=\frac{1}{2}\quad\text{and}\quad\mathbb{E}[Y^2]=\frac{1}{2},$$

and thus

$$\mathrm{Var}(Y)=\mathbb{E}[Y^2]-\mathbb{E}[Y]^2=\frac{1}{4}.$$

At the end

$$\mathrm{Var}(e^3X-\sqrt{5}Y)=\frac{2e^6}{3}+\frac{5}{4}.$$

Problem 4

Let V denote the first shop where Peter forgets his bag. It's a *Geom* $\left(\frac{1}{4}\right)$ r.v.. The probability that he forgets his bag in the last shop given he forgot his back somewhere is given by

$$\mathbb{P}\{V=4 \mid V \leq 4\} = \frac{\mathbb{P}\{V=4, V\leq 4\}}{\mathbb{P}\{V\leq 4\}} = \frac{\mathbb{P}\{V=4\}}{1-\mathbb{P}\{V>4\}} = \frac{\frac{1}{4}(\frac{3}{4})^3}{1-\left(\frac{3}{4}\right)^4} = \frac{27}{175}.$$

Problem 5

What we know is $\mathbb{P}(T) = \mathbb{P}(H) = \frac{1}{2}$, $\mathbb{P}(R_i \mid H) = \frac{2}{3}$, $\mathbb{P}(R_i \mid T) = \frac{1}{3}$, for all $i = \{1,2\}$, $\mathbb{P}(R_2 \mid H \cap R_1) = \frac{2}{3}$ and $\mathbb{P}(R_2 \mid T \cap R_1) = \frac{1}{3}$ (this can be easier to visualize on a tree). We also know more information, but this is all what we need to solve our problem. Not that what following looks a bit barbarous. But just do a tree, and you'll see how simple it is :-) However, at the exam, I need to see those formula (and there justification as I did).

1. Using formula of total probability :

$$\mathbb{P}(R_1) = \mathbb{P}(R_1 \mid H)\mathbb{P}(H) + \mathbb{P}(R_1 \mid T)\mathbb{P}(T) = ...$$

2. Using definition of conditonnal probability (or Bayes works as well),

$$\mathbb{P}(H \mid R_1) = \frac{\mathbb{P}(R_1 \mid H)\mathbb{P}(H)}{\mathbb{P}(R_1)} = ...$$

3. By definition of conditional probability,

$$\mathbb{P}(R_2 \mid R_1) = \frac{\mathbb{P}(R_1 \cap R_2)}{\mathbb{P}(R_1)}.$$

Using formula of total probability,

$$\mathbb{P}(R_1 \cap R_2) = \mathbb{P}(R_1 \cap R_2 \mid H)\mathbb{P}(H) + \mathbb{P}(R_1 \cap R_2 \mid T)\mathbb{P}(T).$$

Using definition of conditional probability

$$\mathbb{P}(R_1 \cap R_2 \mid H) = \mathbb{P}(R_1 \mid H)\mathbb{P}(R_2 \mid H \cap R_1)$$

and

$$\mathbb{P}(R_1 \cap R_2 \mid T) = \mathbb{P}(R_1 \mid T)\mathbb{P}(R_2 \mid T \cap R_1).$$

Combine all will gives the right result.

4. We know $\mathbb{P}(R_1)$ and $\mathbb{P}(R_1 \cap R_2)$ (from previous questions). To compute $\mathbb{P}(R_2)$, we use definition of conditional expectation

$$\mathbb{P}(R_2) = \mathbb{P}(R_2 \mid T)\mathbb{P}(T) + \mathbb{P}(R_2 \mid H)\mathbb{P}(H) = \mathbb{P}(R_1).$$

We you have computed this, and see that $\mathbb{P}(R_1 \cap R_2) \neq \mathbb{P}(R_1)\mathbb{P}(R_2)$. Therefore there are not independent.

Problem 6

1. $\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(\Omega) = 1$ and if $\{A_i\}_{i=1}^{\infty}$ is a disjoints collection of Borel sets, then

$$\mathbb{P}_X\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(X^{-1}\left\{\bigcup_{i=1}^{\infty} A_i\right\}\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty}\{X \in A_i\}\right) \underset{(*)}{=} \sum_{i=1}^{\infty}\mathbb{P}\{X \in A_i\} = \sum_{i=1}^{\infty}\mathbb{P}_X(A_i),$$

where $(*)$ come from the fat that $\mathbb{P}$ is a measure and $X^{-1}(A_i) = \{X \in A_i\}$ are disjoints. Indeed, if $\omega \in X^{-1}(A_i) \cap X^{-1}(A_j) = X^{-1}(A_i \cap A_j)$, then $X(\omega) \in A_i \cap A_j = \emptyset$, which is impossible. Therefore $X^{-1}(A_i) \cap X^{-1}(A_j) = \varnothing$.

2. (a) We have that

$$\mathbb{E}_X[Y] = \int_0^2 e^{-x}\,\mathrm{d}x + \int_1^3 e^{-x}\,\mathrm{d}x = 1 - e^{-2} + e^{-1} - e^{-3},$$

$$\mathbb{E}_X[Z] = \mathbb{P}_X([1,5]) = \int_1^5 e^{-x}\,\mathrm{d}x = e^{-1} - e^{-5}.$$

Moeovr,

$$YZ = (\mathbf{1}_{[0,2]} + \mathbf{1}_{[0,3]})\mathbf{1}_{[0,5]} = \mathbf{1}_{[0,2]}\mathbf{1}_{[1,5]} + \mathbf{1}_{[1,3]}\mathbf{1}_{[1,5]} = \mathbf{1}_{[1,2]} + \mathbf{1}_{[1,3]}.$$

Therefore,

$$\mathbb{E}_X[YZ] = \int_1^2 e^{-x}\,\mathrm{d}x + \int_1^3 e^{-x}\,\mathrm{d}x = e^{-1} - e^{-2} + e^{-1} - e^{-3} = 2e^{-1} - e^{-2} - e^{-3}.$$

(b) We could check that $\mathbb{E}_X[YZ] \neq \mathbb{E}_X[Y]\mathbb{E}_X[Z]$, and thus Y and Z are not independent.

3. (a) We have to prove that $(Z \circ X)^{-1}(B) \in \mathcal{F}$ for all $B \in \mathcal{B}(\mathbb{R})$. Notice that

$$(Z \circ X)^{-1}(B) = X^{-1}(Z^{-1}(B)).$$

Since Z is a random variable on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$, we have that $Z^{-1}(B) \in \mathcal{B}(\mathbb{R})$ for all Borel-set. Also, X is a random variable on $(\Omega, \mathcal{F})$, and thus $X^{-1}(\tilde{B}) \in \mathcal{F}$ for all Borel set $\tilde{B}$. In particular, $X^{-1}(Z^{-1}(B)) \in \mathcal{F}$ for all Borel set $B \in \mathcal{B}(\mathbb{R})$. The claim follow.

(b) One can remark that $Z \circ X = \mathbf{1}_{[1,5]}(X(\omega))$, and thus

$$\mathbb{E}[Z \circ X] = \mathbb{P}\{Z = 1\} = \mathbb{P}\{X \in [1,5]\} = \mathbb{P}_X([1,5]) = \mathbb{E}_X[Z] = e^{-1} - e^{-5}.$$

Problem 7

1. (a) i. Since $(A \setminus B) \cup B$ is dijoint,

$$\mathbb{P}(A) = \mathbb{P}((A \setminus B) \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B),$$

and thus

$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(B).$$

ii. No it's not true. For example, take $\mathbb{P} = Unif[0,1]$, $A = [0,1/2]$ and $B = [1/2,1]$. Then, $\mathbb{P}(A) = \mathbb{P}(B)$, but

$$\mathbb{P}(A \setminus B) = \mathbb{P}([0,1/2)) = \frac{1}{2} \neq \mathbb{P}(A) - \mathbb{P}(B) = 0.$$

(b) Suppose A and B independent. Then,

$$\mathbb{P}(A)\mathbb{P}(B^c) = \mathbb{P}(A)(1-\mathbb{P}(B)) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \underset{(1)}{=} \mathbb{P}(A) - \mathbb{P}(A \cap B) \underset{(2)}{=} \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A \cap B^c),$$

where (1) come from the independence of A and B and (2) come from the fact that $A \cap B \subset A$ and the previous question.

2. Set $E = \bigcap_{n \in \mathbb{N}} B_n$, and $E_n = B_n \setminus B_{n+1}$. Notice that the E_n's are disjoint and E is also disjoint with the E_n. Now

$$B_1 = E \cup \bigcup_{i=1}^{\infty} E_n,$$

and thus,

$$\mathbb{P}(B_1) = \mathbb{P}(E) + \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}(E) + \sum_{k=1}^{\infty} \mathbb{P}(B_k \setminus B_{k+1}). \tag{1}$$

Now, using **1.**(a)i. of this exercise,

$$\sum_{k=1}^{n-1} \mathbb{P}(B_k \setminus B_{k+1}) = \sum_{k=1}^{n} \mathbb{P}(B_k) - \mathbb{P}(B_{k+1}) = \mathbb{P}(B_1) - \mathbb{P}(B_n).$$

Taking the limit $n \to \infty$ and replacing in (1), we finally obtain

$$\mathbb{P}(E) + \mathbb{P}(B_1) - \lim_{n\to\infty} \mathbb{P}(B_n) = \mathbb{P}(B_1),$$

and thus

$$\mathbb{P}(E) = \lim_{n\to\infty} \mathbb{P}(B_n),$$

as wished.

3. This was the most difficult exercise of this exam. Fix $n \in \mathbb{N}$. Since $\mathbb{P}\{X_n \leq X_{n+1}\} = 1$, there is a set $\mathcal{N}_n$ s.t. $\mathbb{P}(\mathcal{N}_n) = 0$ and for all $\omega \notin \mathcal{N}_n$, $X_n(\omega) \leq X_{n+1}(\omega)$. Set $K = \bigcup_{n\in\mathbb{N}} \mathcal{N}_n$. Set $B_n = \{\omega \in K^c \mid X_n \leq x\}$ and $O_n = \{\omega \in K \mid X_n \leq x\}$. In particular, $B_{n+1} \subset B_n$ and

$$\bigcap_{n\in\mathbb{N}} \{X_n \leq x\} = \bigcap_{n\in\mathbb{N}} (B_n \cup O_n) = \bigcap_{n\in\mathbb{N}} B_n \cup \bigcap_{n\in\mathbb{N}} O_n.$$

Now, these two sets are disjoint, and $\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} O_n\right) = 0$. Therefore

$$\mathbb{P}\left(\bigcap_{n\in\mathbb{N}} \{X_n \leq x\}\right) = \mathbb{P}\left(\bigcap_{n\in\mathbb{N}} B_n\right) \underset{(1)}{=} \lim_{n\to\infty} \mathbb{P}(B_n) \underset{(2)}{=} \lim_{n\to\infty} \mathbb{P}(B_n \cup O_n) = \lim_{n\to\infty} \mathbb{P}\{X_n \leq x\},$$

where (1) come from **2.** and (2) comes from the fact that $\mathbb{P}(O_n) = 0$ for all n and $\mathbb{P}(O_n) + \mathbb{P}(B_n) = \mathbb{P}(B_n \cup O_n)$ because B_n and O_n are disjoints.

Problem 8

1. (a) Ω is the set of all couple that got married in 2018, $\mathcal{F}$ is the power set and $\mathbb{P}\{\omega\} = \frac{1}{365^2} =: p$.

(b) If X is the number of couple that were born that 24th of january, then

$$\mathbb{P}\{X = 25\} = \binom{40000}{5} p^5 (1-p)^{40000-5}.$$

This can be approximated by a Poisson law of parameter $\lambda = 40000p$. I.e.

$$\mathbb{P}\{X = 25\} = e^{-\lambda} \frac{\lambda^{25}}{(25)!}.$$

2. (a) Same but $\mathbb{P}\{\omega\} = \frac{1}{365} =: q$.

(b) Same with $\lambda = 40000q$.