

## Probability : Practice Exam (solution)

### Problem 1

1. See your notes.
2. Let  $\{A_i\}_{i=1}^{\infty}$  a collection of  $\mathcal{F}$ . Since  $\mathcal{F}$  is a  $\sigma$  algebra, it's stable by complementary and countable union. Therefore,

$$\bigcap_{i=1}^{\infty} A_i = \left( \bigcup_{i=1}^{\infty} A_i^c \right)^c \in \mathcal{F}.$$

3. (a) Let prove that  $\mathcal{F}_B$  is a  $\sigma$ -algebra. The fact that  $B$  and  $\emptyset$  are in  $\mathcal{F}_B$  is clear. Let  $A \in \mathcal{F}_B$ . i.e. there is  $C \in \mathcal{F}$  s.t.  $A = C \cap B$ . Then,

$$A^c = B \setminus A = B \setminus (C \cap B) = (B \cap C^c) \cup (B \cap B^c) = B \cap C^c.$$

Here  $C^c = \Omega \setminus C \in \mathcal{F}$ . Therefore,  $A^c \in \mathcal{F}_B$ . Let  $\{A_i\}_{i=1}^{\infty}$  a collection of element of  $\mathcal{F}_B$ . I.e. for all  $A_i$  there is  $C_i \in \mathcal{F}$  s.t.  $A_i = B \cap C_i$ . Then,

$$\bigcup_{i=1}^{\infty} A_i = \bigcup_{i=1}^{\infty} (C_i \cap B) = B \cap \underbrace{\left( \bigcup_{i=1}^{\infty} C_i \right)}_{\in \mathcal{F}} \in \mathcal{F}_B.$$

Let show that  $\mathbb{Q}$  is a probability on  $(B, \mathcal{F}_B)$ . We have  $\mathbb{Q}(B) = \frac{\mathbb{P}(B)}{\mathbb{P}(B)} = 1$ . Also, if  $\{A_i\}_{i=1}^{\infty}$  is disjoint collection of  $\mathcal{F}_B$ , then

$$\mathbb{Q} \left( \bigcup_{i=1}^{\infty} A_i \right) = \frac{1}{\mathbb{P}(B)} \mathbb{P} \left( \bigcup_{i=1}^{\infty} A_i \right) \stackrel{(*)}{=} \sum_{i=1}^{\infty} \frac{\mathbb{P}(A_i)}{\mathbb{P}(B)} = \sum_{i=1}^{\infty} \mathbb{Q}(A_i),$$

where  $(*)$  come from the fact that  $\mathbb{P}$  is a probability on  $(\Omega, \mathbb{F})$  and  $A_i \in \mathcal{F}$  for all  $i$ .

- (b) i. It's not a random variable because  $X^{-1}(\{1\}) = \{(P, F)\} \notin \mathcal{F}$ .
- ii. It is a random variable because  $X|_B = 0$  and thus

$$X^{-1}(B) = \begin{cases} \emptyset & 0 \notin B \\ B & 0 \in B. \end{cases}$$

Therefore  $X^{-1}(B) \in \mathcal{F}_B$  for all Borel set  $B$ .

### Problem 2

Let  $X_i$  denote « The  $i^{\text{th}}$  client comes ». Then  $X_i \sim \text{Bern}(0.8)$ . Set

$$S_n = X_1 + \dots + X_n.$$

Since the  $X_i$  are i.i.d., we have that  $S_n \sim \text{Binom}(n, 0.8)$ .

1. Using the DeMoivre Laplace theorem

$$\mathbb{P}\{S_{190} \leq 150\} = \mathbb{P}\left\{ \frac{S_{190} - 190 \cdot 0.8}{\sqrt{190 \cdot 0.2 \cdot 0.8}} \leq \frac{150 - 190 \cdot 0.8}{\sqrt{190 \cdot 0.2 \cdot 0.8}} \right\} = \Phi_{0,1}(-0.36) \approx 0.36.$$

2. We have to find  $n$  s.t.  $\mathbb{P}\{S_n \leq 150\} \geq 0.99$ . This equation is equivalent to

$$\mathbb{P}\left\{ \frac{S_n - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}} \leq \frac{150 - n \cdot 0.8}{\sqrt{n \cdot 0.8 \cdot 0.2}} \right\} = \Phi_{0,1}(2.33) \iff \frac{150 - 0.8n}{\sqrt{n \cdot 0.2 \cdot 0.8}} \geq 2.33 \iff n \leq 172.$$

So, the restaurant can take at most 172 reservation.

### Problem 3

1. See your notes.

2. (a)

$$\begin{aligned}\text{Var}(aX + bY) &= \mathbb{E}\left[(aX + bY - \mathbb{E}[aX + bY])^2\right] \\ &= \mathbb{E}\left[((aX - \mathbb{E}[aX]) + (bY - \mathbb{E}[bY]))^2\right] \\ &= \mathbb{E}\left[(aX - \mathbb{E}[aX])^2\right] + 2\mathbb{E}\left[(aX - \mathbb{E}[aX])(bY - \mathbb{E}[bY])\right] + \mathbb{E}\left[(bY - \mathbb{E}[bY])^2\right] \\ &= a^2\mathbb{E}\left[(X - \mathbb{E}[X])^2\right] + 2ab\mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] + b^2\mathbb{E}\left[(Y - \mathbb{E}[Y])^2\right] \\ &= a^2\text{Var}(X) + b^2\text{Var}(Y) + 2ab\text{Cov}(X, Y).\end{aligned}$$

(b) If  $X$  and  $Y$  are independent, we know that  $\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$ , and thus

$$\begin{aligned}\text{Cov}(X, Y) &= \mathbb{E}\left[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])\right] \\ &= \mathbb{E}\left[XY - X\mathbb{E}[Y] - Y\mathbb{E}[X] + \mathbb{E}[X]\mathbb{E}[Y]\right] \\ &= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y] \\ &= 0.\end{aligned}$$

3. (a) We have

$$\begin{aligned}\mathbb{P}\{X = 0, Y = 0\} &= \mathbb{P}\{1\} = \frac{1}{6} = \mathbb{P}\{X = 0\}\mathbb{P}\{Y = 0\}, \\ \mathbb{P}\{X = 1, Y = 0\} &= \mathbb{P}\{3\} = \frac{1}{6} = \mathbb{P}\{X = 1\}\mathbb{P}\{Y = 0\} \\ \mathbb{P}\{X = 2, Y = 0\} &= \mathbb{P}\{5\} = \frac{1}{6} = \mathbb{P}\{X = 2\}\mathbb{P}\{Y = 0\}.\end{aligned}$$

Notice that since  $A$  and  $B$  independent implies that  $A$  and  $B^c$  are independent (proved in problem 6 of the sheet 2 and also is problem 6 of this exam), we can conclude that  $X$  and  $Y$  are independent.

(b) Using 2. and the fact that  $X$  and  $Y$  are independent, we get

$$\text{Var}(e^3X - \sqrt{5}Y) = e^6\text{Var}(X) + 5\text{Var}(X).$$

Now,

$$\begin{aligned}\mathbb{E}[X] &= \mathbb{P}\{3, 4\} + 2\mathbb{P}\{5, 6\} = 1, \\ \mathbb{E}[X^2] &= \mathbb{P}\{3, 4\} + 4\mathbb{P}\{5, 6\} = \frac{5}{3},\end{aligned}$$

and thus

$$\text{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \frac{2}{3}.$$

Also

$$\mathbb{E}[Y] = \frac{1}{2} \quad \text{and} \quad \mathbb{E}[Y^2] = \frac{1}{2},$$

and thus

$$\text{Var}(Y) = \mathbb{E}[Y^2] - \mathbb{E}[Y]^2 = \frac{1}{4}.$$

At the end

$$\text{Var}(e^3X - \sqrt{5}Y) = \frac{2e^6}{3} + \frac{5}{4}.$$

## Problem 4

Let  $V$  denote the first shop where Peter forgets his bag. It's a  $Geom(\frac{1}{4})$  r.v.. The probability that he forgets his bag in the last shop given he forgot his back somewhere is given by

$$\mathbb{P}\{V = 4 \mid V \leq 4\} = \frac{\mathbb{P}\{V = 4, V \leq 4\}}{\mathbb{P}\{V \leq 4\}} = \frac{\mathbb{P}\{V = 4\}}{1 - \mathbb{P}\{V > 4\}} = \frac{\frac{1}{4}(\frac{3}{4})^3}{1 - (\frac{3}{4})^4} = \frac{27}{175}.$$

## Problem 5

What we know is  $\mathbb{P}(T) = \mathbb{P}(H) = \frac{1}{2}$ ,  $\mathbb{P}(R_i \mid H) = \frac{2}{3}$ ,  $\mathbb{P}(R_i \mid T) = \frac{1}{3}$ , for all  $i = \{1, 2\}$ ,  $\mathbb{P}(R_2 \mid H \cap R_1) = \frac{2}{3}$  and  $\mathbb{P}(R_2 \mid T \cap R_1) = \frac{1}{3}$  (this can be easier to visualize on a tree). We also know more information, but this is all what we need to solve our problem. Not that what following looks a bit barbarous. But just do a tree, and you'll see how simple it is :-). However, at the exam, I need to see those formula (and there justification as I did).

1. Using formula of total probability :

$$\mathbb{P}(R_1) = \mathbb{P}(R_1 \mid H)\mathbb{P}(H) + \mathbb{P}(R_1 \mid T)\mathbb{P}(T) = \dots$$

2. Using definition of conditonnal probability (or Bayes works as well),

$$\mathbb{P}(H \mid R_1) = \frac{\mathbb{P}(R_1 \mid H)\mathbb{P}(H)}{\mathbb{P}(R_1)} = \dots$$

3. By definition of conditional probability,

$$\mathbb{P}(R_2 \mid R_1) = \frac{\mathbb{P}(R_1 \cap R_2)}{\mathbb{P}(R_1)}.$$

Using formula of total probability,

$$\mathbb{P}(R_1 \cap R_2) = \mathbb{P}(R_1 \cap R_2 \mid H)\mathbb{P}(H) + \mathbb{P}(R_1 \cap R_2 \mid T)\mathbb{P}(T).$$

Using definition of conditional probability

$$\mathbb{P}(R_1 \cap R_2 \mid H) = \mathbb{P}(R_1 \mid H)\mathbb{P}(R_2 \mid H \cap R_1)$$

and

$$\mathbb{P}(R_1 \cap R_2 \mid T) = \mathbb{P}(R_1 \mid T)\mathbb{P}(R_2 \mid T \cap R_1).$$

Combine all will gives the right result.

4. We know  $\mathbb{P}(R_1)$  and  $\mathbb{P}(R_1 \cap R_2)$  (from previous questions). To compute  $\mathbb{P}(R_2)$ , we use definition of conditional expectation

$$\mathbb{P}(R_2) = \mathbb{P}(R_2 \mid T)\mathbb{P}(T) + \mathbb{P}(R_2 \mid H)\mathbb{P}(H) = \mathbb{P}(R_1).$$

We you have computed this, and see that  $\mathbb{P}(R_1 \cap R_2) \neq \mathbb{P}(R_1)\mathbb{P}(R_2)$ . Therefore there are not independent.

## Problem 6

1.  $\mathbb{P}_X(\mathbb{R}) = \mathbb{P}(\Omega) = 1$  and if  $\{A_i\}_{i=1}^{\infty}$  is a disjoints collection of Borel sets, then

$$\mathbb{P}_X\left(\bigcup_{i=1}^{\infty} A_i\right) = \mathbb{P}\left(X^{-1}\left(\bigcup_{i=1}^{\infty} A_i\right)\right) = \mathbb{P}\left(\bigcup_{i=1}^{\infty} \{X \in A_i\}\right) \stackrel{(*)}{=} \sum_{i=1}^{\infty} \mathbb{P}\{X \in A_i\} = \sum_{i=1}^{\infty} \mathbb{P}_X(A_i),$$

where  $(*)$  come from the fat that  $\mathbb{P}$  is a measure and the  $X^{-1}(A_i) = \{X \in A_i\}$ 's are disjoints. Indeed, if  $\omega \in X^{-1}(A_i) \cap X^{-1}(A_j) = X^{-1}(A_i \cap A_j)$ , then  $X(\omega) \in A_i \cap A_j = \emptyset$ , which is impossible. Therefore  $X^{-1}(A_i) \cap X^{-1}(A_j) = \emptyset$ .

2. (a) We have that

$$\mathbb{E}_X[Y] = \int_0^2 e^{-x} dx + \int_1^3 e^{-x} dx = 1 - e^{-2} + e^{-1} - e^{-3},$$

$$\mathbb{E}_X[Z] = \mathbb{P}_X([1, 5]) = \int_1^5 e^{-x} dx = e^{-1} - e^{-5}.$$

Moreover,

$$YZ = (\mathbf{1}_{[0,2]} + \mathbf{1}_{[0,3]})\mathbf{1}_{[0,5]} = \mathbf{1}_{[0,2]}\mathbf{1}_{[1,5]} + \mathbf{1}_{[1,3]}\mathbf{1}_{[1,5]} = \mathbf{1}_{[1,2]} + \mathbf{1}_{[1,3]}.$$

Therefore,

$$\mathbb{E}_X[YZ] = \int_1^2 e^{-x} dx + \int_1^3 e^{-x} dx = e^{-1} - e^{-2} + e^{-1} - e^{-3} = 2e^{-1} - e^{-2} - e^{-3}.$$

- (b) We could check that  $\mathbb{E}_X[YZ] \neq \mathbb{E}_X[Y]\mathbb{E}_X[Z]$ , and thus  $Y$  and  $Z$  are not independent.

3. (a) We have to prove that  $(Z \circ X)^{-1}(B) \in \mathcal{F}$  for all  $B \in \mathcal{B}(\mathbb{R})$ . Notice that

$$(Z \circ X)^{-1}(B) = X^{-1}(Z^{-1}(B)).$$

Since  $Z$  is a random variable on  $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ , we have that  $Z^{-1}(B) \in \mathcal{B}(\mathbb{R})$  for all Borel-set. Also,  $X$  is a random variable on  $(\Omega, \mathcal{F})$ , and thus  $X^{-1}(\tilde{B}) \in \mathcal{F}$  for all Borel set  $\tilde{B}$ . In particular,  $X^{-1}(Z^{-1}(B)) \in \mathcal{F}$  for all Borel set  $B \in \mathcal{B}(\mathbb{R})$ . The claim follows.

- (b) One can remark that  $Z \circ X = \mathbf{1}_{[1,5]}(X(\omega))$ , and thus

$$\mathbb{E}[Z \circ X] = \mathbb{P}\{Z = 1\} = \mathbb{P}\{X \in [1, 5]\} = \mathbb{P}_X([1, 5]) = \mathbb{E}_X[Z] = e^{-1} - e^{-5}.$$

## Problem 7

1. (a) i. Since  $(A \setminus B) \cup B$  is disjoint,

$$\mathbb{P}(A) = \mathbb{P}((A \setminus B) \cup B) = \mathbb{P}(A \setminus B) + \mathbb{P}(B),$$

and thus

$$\mathbb{P}(A \setminus B) = \mathbb{P}(A) - \mathbb{P}(B).$$

- ii. No it's not true. For example, take  $\mathbb{P} = \text{Unif}[0, 1]$ ,  $A = [0, 1/2]$  and  $B = [1/2, 1]$ . Then,  $\mathbb{P}(A) = \mathbb{P}(B)$ , but

$$\mathbb{P}(A \setminus B) = \mathbb{P}([0, 1/2)) = \frac{1}{2} \neq \mathbb{P}(A) - \mathbb{P}(B) = 0.$$

- (b) Suppose  $A$  and  $B$  independent. Then,

$$\mathbb{P}(A)\mathbb{P}(B^c) = \mathbb{P}(A)(1 - \mathbb{P}(B)) = \mathbb{P}(A) - \mathbb{P}(A)\mathbb{P}(B) \stackrel{(1)}{=} \mathbb{P}(A) - \mathbb{P}(A \cap B) \stackrel{(2)}{=} \mathbb{P}(A \setminus (A \cap B)) = \mathbb{P}(A \cap B^c),$$

where (1) come from the independence of  $A$  and  $B$  and (2) come from the fact that  $A \cap B \subset A$  and the previous question.

2. Set  $E = \bigcap_{n \in \mathbb{N}} B_n$ , and  $E_n = B_n \setminus B_{n+1}$ . Notice that the  $E_n$ 's are disjoint and  $E$  is also disjoint with the  $E_n$ . Now

$$B_1 = E \cup \bigcup_{i=1}^{\infty} E_n,$$

and thus,

$$\mathbb{P}(B_1) = \mathbb{P}(E) + \mathbb{P}\left(\bigcup_{i=1}^{\infty} E_i\right) = \mathbb{P}(E) + \sum_{k=1}^{\infty} \mathbb{P}(B_k \setminus B_{k+1}). \quad (1)$$

Now, using 1.(a)i. of this exercise,

$$\sum_{k=1}^{n-1} \mathbb{P}(B_k \setminus B_{k+1}) = \sum_{k=1}^n \mathbb{P}(B_k) - \mathbb{P}(B_{k+1}) = \mathbb{P}(B_1) - \mathbb{P}(B_n).$$

Taking the limit  $n \rightarrow \infty$  and replacing in (1), we finally obtain

$$\mathbb{P}(E) + \mathbb{P}(B_1) - \lim_{n \rightarrow \infty} \mathbb{P}(B_n) = \mathbb{P}(B_1),$$

and thus

$$\mathbb{P}(E) = \lim_{n \rightarrow \infty} \mathbb{P}(B_n),$$

as wished.

- 3. This was the most difficult exercise of this exam.** Fix  $n \in \mathbb{N}$ . Since  $\mathbb{P}\{X_n \leq X_{n+1}\} = 1$ , there is a set  $\mathcal{N}_n$  s.t.  $\mathbb{P}(\mathcal{N}_n) = 0$  and for all  $\omega \notin \mathcal{N}_n$ ,  $X_n(\omega) \leq X_{n+1}(\omega)$ . Set  $K = \bigcup_{n \in \mathbb{N}} \mathcal{N}_n$ . Set  $B_n = \{\omega \in K^c \mid X_n \leq x\}$  and  $O_n = \{\omega \in K \mid X_n \leq x\}$ . In particular,  $B_{n+1} \subset B_n$  and

$$\bigcap_{n \in \mathbb{N}} \{X_n \leq x\} = \bigcap_{n \in \mathbb{N}} (B_n \cup O_n) = \bigcap_{n \in \mathbb{N}} B_n \cup \bigcap_{n \in \mathbb{N}} O_n.$$

Now, these two sets are disjoint, and  $\mathbb{P}\left(\bigcap_{n \in \mathbb{N}} O_n\right) = 0$ . Therefore

$$\mathbb{P}\left(\bigcap_{n \in \mathbb{N}} \{X_n \leq x\}\right) = \mathbb{P}\left(\bigcap_{n \in \mathbb{N}} B_n\right) \stackrel{(1)}{=} \lim_{n \rightarrow \infty} \mathbb{P}(B_n) \stackrel{(2)}{=} \lim_{n \rightarrow \infty} \mathbb{P}(B_n \cup O_n) = \lim_{n \rightarrow \infty} \mathbb{P}\{X_n \leq x\},$$

where (1) come from 2. and (2) comes from the fact that  $\mathbb{P}(O_n) = 0$  for all  $n$  and  $\mathbb{P}(O_n) + \mathbb{P}(B_n) = \mathbb{P}(B_n \cup O_n)$  because  $B_n$  and  $O_n$  are disjoint.

## Problem 8

1. (a)  $\Omega$  is the set of all couple that got married in 2018,  $\mathcal{F}$  is the power set and  $\mathbb{P}\{\omega\} = \frac{1}{365^2} =: p$ .
- (b) If  $X$  is the number of couple that were born that 24th of january, then

$$\mathbb{P}\{X = 25\} = \binom{40000}{5} p^5 (1-p)^{40000-5}.$$

This can be approximated by a Poisson law of parameter  $\lambda = 40000p$ . I.e.

$$\mathbb{P}\{X = 25\} = e^{-\lambda} \frac{\lambda^{25}}{(25)!}.$$

2. (a) Same but  $\mathbb{P}\{\omega\} = \frac{1}{365} =: q$ .
- (b) Same with  $\lambda = 40000q$ .