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Probability : Sheet 7 (solution)

Problem 1

$$\mathbb{Q}(\{x\}) = \mathbb{Q}((-\infty, x] \setminus (-\infty, x)) = \mathbb{Q}((-\infty, x]) - \mathbb{Q}((-\infty, x)).$$

Set $\psi(y) = \mathbb{Q}((-\infty, y])$. By a proposition of the couse, ψ is continuous. Then,

$$(-\infty, x) = \bigcup_{n\geq 1} \left(-\infty, x - \frac{1}{n}\right],$$

and thus, by continuity of the measure,

$$\mathbb{Q}((-\infty, x)) = \mathbb{Q}\left(\bigcup_{n\geq 1} \left(-\infty, x - \frac{1}{n}\right]\right) = \lim_{n\to\infty} \mathbb{Q}\left(\left(-\infty, x - \frac{1}{n}\right]\right) = \lim_{n\to\infty} \psi\left(x - \frac{1}{n}\right) \underset{(*)}{=} \psi(x) = \mathbb{Q}((-\infty, x]),$$

where $(*)$ come from continuity of ψ. Therefore,

$$\mathbb{Q}((-\infty, x]) = \mathbb{Q}((-\infty, x)),$$

and thus $\mathbb{Q}(\{x\}) = 0$.

Problem 2

Since $\psi \geq 0$, it define a density function if and only if

$$\int_{\mathbb{R}} \psi(x)\,\mathrm{d}x = 1,$$

i.e. if and only if $\lambda = \frac{1}{1000}$.

$$\mathbb{P}\{50 \leq X \leq 150\} = \int_{50}^{150} \psi(x)\,\mathrm{d}x = e^{-0.05} - e^{-0.15}.$$

Problem 3

Let f_X the density function of X. Since X i absolutely continuous, there is f_X s.t.

$$\mathbb{P}\{X \leq y\} = \int_{-\infty}^{y} f_X(x)\,\mathrm{d}x.$$

Let first derive the density function of $aX + b$;

$$\mathbb{P}\{aX + b \leq x\} = \mathbb{P}\left\{X \leq \frac{x-b}{a}\right\} = \int_{-\infty}^{\frac{x-b}{a}} f_X(x)\,\mathrm{d}x,$$

and thus,

$$f_{aX+b}(x) = \frac{\mathrm{d}}{\mathrm{d}x}\mathbb{P}\{aX + b \leq x\} = \frac{1}{a} f_X\left(\frac{x-b}{a}\right).$$

Therefore,

$$\mathbb{E}[aX+b] = \int_{\mathbb{R}} x f_{aX+b}(x)\,\mathrm{d}x = \frac{1}{a}\int_{\mathbb{R}} x f_X\left(\frac{x-b}{a}\right)\,\mathrm{d}x \underset{u=\frac{x-b}{a}}{=} \int_{\mathbb{R}} (au+b) f_X(u)\,\mathrm{d}u$$

$$= a\int_{\mathbb{R}} u f_X(u)\,\mathrm{d}u + b\underbrace{\int_{\mathbb{R}} f_X(u)\,\mathrm{d}u}_{=1} = a\mathbb{E}[X]+b.$$

Problem 7

Recall that a r.v. X is $Par_{t_n,\alpha}$ distributed if it cumulative function is

$$\mathbb{P}\{X\le x\} = F_X(x) = \begin{cases} 1-\left(\frac{t_n}{x}\right)^{\alpha} & x\ge t_n \\ 0 & \text{otherwise}\end{cases}$$

Set $Y=\log\left(\frac{X}{t_n}\right)$ and suppose $X\sim Par_{t_n,\alpha}$. Then,

$$\mathbb{P}\{Y\le y\} = \mathbb{P}\left\{\log\left(\frac{X}{t_n}\right)\le y\right\} = \mathbb{P}\left\{X\le t_n e^y\right\} = 1-\left(\frac{t_n}{t_n e^y}\right)^{\alpha} = 1-e^{-\alpha y}.$$

and thus $Y\sim\exp(\alpha)$. Conversely, if $Y\sim\exp(\alpha)$, then

$$\mathbb{P}\{X\le x\} = \mathbb{P}\left\{\log\left(\frac{X}{t_n}\right)\le\log\left(\frac{x}{t_n}\right)\right\} = \mathbb{P}\left\{Y\le\log\left(\frac{x}{t_n}\right)\right\} = 1-e^{-\alpha\log\left(\frac{x}{t_n}\right)} = 1-\left(\frac{t_n}{x}\right)^{\alpha},$$

and thus $X\sim Par_{t_n,\alpha}$.

Problem 9

An elementary random variable is a linear combinaison of finitely many unitary function of disjoints events, i.e. if $(\Omega,\mathcal{F},\mathbb{P})$ is a probability space, an elementary r.v. is a function of the form

$$a_1\mathbf{1}_{A_1}+\ldots+a_n\mathbf{1}_{A_n},$$

for $A_i\in\mathcal{F}$ for all i and all A_i's are disjoints. The integral integral of such a function is given by

$$\int_{\Omega}(a_1\mathbf{1}_{A_1}+\ldots+a_n\mathbf{1}_{A_n})\,\mathrm{d}\mathbb{P} = a_1\mathbb{P}(A_1)+\ldots+a_n\mathbb{P}(A_n).$$

As you can see, the integral of a r.v. correspond to its expectation.

Problem 10

1.

$$\int_{\Omega} X\,\mathrm{d}\mathbb{P} = 1\mathbb{P}([0,1]) - 1\mathbb{P}([-1,0]) + 1\cdot\mathbb{P}(\{0\}).$$

Using

$$\mathbb{P}(A) = \frac{1}{\sqrt{2\pi}}\int_A e^{-\frac{x^2}{2}}\,\mathrm{d}x,$$

the claim follow (calculation are left to the readers).

Remark : As you can see, in the previous integral, I didn't right the elementary r.v. as a sum of unitary function of disjoints events. The reason is that the integral doesn't depend on the writing, i.e. let

$$X=\sum_{i=1}^{n} a_i\mathbf{1}_{A_i},$$

where A_j's are not necessarily disjoints. Write X as a linear combinaison of unitary function of disjoints events, i.e.

$$X = \sum_{i=1}^{m} b_j \mathbf{1}_{B_j},$$

where the B_j's are disjoints. Then,

$$\int_\Omega X \,\mathrm{d}P = \sum_{i=1}^{n} a_i \mathbb{P}(A_i) = \sum_{i=1}^{m} b_i \mathbb{P}(B_i).$$

2.

$$\int_\Omega X \,\mathrm{d}\mathbb{P} = 10 \cdot \mathbb{P}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) + 5 \cdot \mathbb{P}([1, 10]) + 10 \cdot \mathbb{P}(\{100\}).$$

Now,

$$\mathbb{P}(A) = \sum_{k \in \mathbb{N} \cap A} \mathbb{P}(\{k\}) = \sum_{k \in \mathbb{N} \cap A} 2^{-k}.$$

Therefore

$$\int_\Omega X \,\mathrm{d}\mathbb{P} = 5 \cdot \sum_{k=1}^{10} 2^{-k} + 10 \cdot 2^{-100} = 5 + 10 \cdot 2^{100} - 5 \cdot 2^{-10}.$$

3. We have that

$$\mathbb{P}(A) = \frac{1}{2}\left(\int_A \mathbf{1}_{[0,1]}(x) \,\mathrm{d}x + \delta_{\frac{1}{2}}(A)\right),$$

where

$$\delta_{\frac{1}{2}}(A) = \begin{cases} 1 & \frac{1}{2} \in A \\ 0 & \frac{1}{2} \notin A \end{cases}.$$

Therefore,

$$\begin{aligned} \int_\Omega X \,\mathrm{d}\mathbb{P} &= \mathbb{P}\left(\left[0, \frac{1}{4}\right]\right) + 2 \cdot \mathbb{P}\left(\left\{\frac{1}{2}\right\}\right) + \mathbb{P}\left(\left[\frac{1}{2}, 1\right]\right) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2}\left(\frac{1}{2} - 1 + 1\right) = \frac{11}{8}. \end{aligned}$$