

## Probability : Sheet 7 (solution)

### Problem 1

$$\mathbb{Q}(\{x\}) = \mathbb{Q}((-\infty, x] \setminus (-\infty, x)) = \mathbb{Q}((-\infty, x]) - \mathbb{Q}((-\infty, x)).$$

Set  $\psi(y) = \mathbb{Q}((-\infty, y])$ . By a proposition of the course,  $\psi$  is continuous. Then,

$$(-\infty, x) = \bigcup_{n \geq 1} \left( -\infty, x - \frac{1}{n} \right],$$

and thus, by continuity of the measure,

$$\mathbb{Q}((-\infty, x)) = \mathbb{Q} \left( \bigcup_{n \geq 1} \left( -\infty, x - \frac{1}{n} \right] \right) = \lim_{n \rightarrow \infty} \mathbb{Q} \left( \left( -\infty, x - \frac{1}{n} \right] \right) = \lim_{n \rightarrow \infty} \psi \left( x - \frac{1}{n} \right) \stackrel{(*)}{=} \psi(x) = \mathbb{Q}((-\infty, x]),$$

where (\*) come from continuity of  $\psi$ . Therefore,

$$\mathbb{Q}((-\infty, x]) = \mathbb{Q}((-\infty, x)),$$

and thus  $\mathbb{Q}(\{x\}) = 0$ .

### Problem 2

Since  $\psi \geq 0$ , it define a density function if and only if

$$\int_{\mathbb{R}} \psi(x) dx = 1,$$

i.e. if and only if  $\lambda = \frac{1}{1000}$ .

$$\mathbb{P}\{50 \leq X \leq 150\} = \int_{50}^{150} \psi(x) dx = e^{-0.05} - e^{-0.15}.$$

### Problem 3

Let  $f_X$  the density function of  $X$ . Since  $X$  is absolutely continuous, there is  $f_X$  s.t.

$$\mathbb{P}\{X \leq y\} = \int_{-\infty}^y f_X(x) dx.$$

Let first derive the density function of  $aX + b$ ;

$$\mathbb{P}\{aX + b \leq x\} = \mathbb{P}\left\{X \leq \frac{x-b}{a}\right\} = \int_{-\infty}^{\frac{x-b}{a}} f_X(x) dx,$$

and thus,

$$f_{aX+b}(x) = \frac{d}{dx} \mathbb{P}\{aX + b \leq x\} = \frac{1}{a} f_X \left( \frac{x-b}{a} \right).$$

Therefore,

$$\begin{aligned}\mathbb{E}[aX + b] &= \int_{\mathbb{R}} xf_{aX+b}(x) dx = \frac{1}{a} \int_{\mathbb{R}} xf_X\left(\frac{x-b}{a}\right) dx \stackrel{u=\frac{x-b}{a}}{=} \int_{\mathbb{R}} (au+b)f_X(u) du \\ &= a \int_{\mathbb{R}} uf_X(u) du + b \underbrace{\int_{\mathbb{R}} f_X(u) du}_{=1} = a\mathbb{E}[X] + b.\end{aligned}$$

### Problem 7

Recall that a r.v.  $X$  is  $Par_{t_n, \alpha}$  distributed if its cumulative function is

$$\mathbb{P}\{X \leq x\} = F_X(x) = \begin{cases} 1 - \left(\frac{t_n}{x}\right)^\alpha & x \geq t_n \\ 0 & \text{otherwise} \end{cases}$$

Set  $Y = \log\left(\frac{X}{t_n}\right)$  and suppose  $X \sim Par_{t_n, \alpha}$ . Then,

$$\mathbb{P}\{Y \leq y\} = \mathbb{P}\left\{\log\left(\frac{X}{t_n}\right) \leq y\right\} = \mathbb{P}\{X \leq t_n e^y\} = 1 - \left(\frac{t_n}{t_n e^y}\right)^\alpha = 1 - e^{-\alpha y}.$$

and thus  $Y \sim \exp(\alpha)$ . Conversely, if  $Y \sim \exp(\alpha)$ , then

$$\mathbb{P}\{X \leq x\} = \mathbb{P}\left\{\log\left(\frac{X}{t_n}\right) \leq \log\left(\frac{x}{t_n}\right)\right\} = \mathbb{P}\left\{Y \leq \log\left(\frac{x}{t_n}\right)\right\} = 1 - e^{-\alpha \log\left(\frac{x}{t_n}\right)} = 1 - \left(\frac{t_n}{x}\right)^\alpha,$$

and thus  $X \sim Par_{t_n, \alpha}$ .

### Problem 9

An elementary random variable is a linear combination of finitely many unitary functions of disjoint events, i.e. if  $(\Omega, \mathcal{F}, \mathbb{P})$  is a probability space, an elementary r.v. is a function of the form

$$a_1 \mathbf{1}_{A_1} + \dots + a_n \mathbf{1}_{A_n},$$

for  $A_i \in \mathcal{F}$  for all  $i$  and all  $A_i$ 's are disjoint. The integral of such a function is given by

$$\int_{\Omega} (a_1 \mathbf{1}_{A_1} + \dots + a_n \mathbf{1}_{A_n}) d\mathbb{P} = a_1 \mathbb{P}(A_1) + \dots + a_n \mathbb{P}(A_n).$$

As you can see, the integral of a r.v. corresponds to its expectation.

### Problem 10

1.

$$\int_{\Omega} X d\mathbb{P} = 1 \cdot \mathbb{P}([0, 1]) - 1 \cdot \mathbb{P}([-1, 0]) + 1 \cdot \mathbb{P}(\{0\}).$$

Using

$$\mathbb{P}(A) = \frac{1}{\sqrt{2\pi}} \int_A e^{-\frac{x^2}{2}} dx,$$

the claim follows (calculations are left to the readers).

**Remark :** As you can see, in the previous integral, I didn't write the elementary r.v. as a sum of unitary functions of disjoint events. The reason is that the integral doesn't depend on the writing, i.e. let

$$X = \sum_{i=1}^n a_i \mathbf{1}_{A_i},$$

where  $A_j$ 's are not necessarily disjoint. Write  $X$  as a linear combination of indicator functions of disjoint events, i.e.

$$X = \sum_{i=1}^m b_i \mathbf{1}_{B_i},$$

where the  $B_j$ 's are disjoint. Then,

$$\int_{\Omega} X \, dP = \sum_{i=1}^n a_i \mathbb{P}(A_i) = \sum_{i=1}^m b_i \mathbb{P}(B_i).$$

**2.**

$$\int_{\Omega} X \, dP = 10 \cdot \mathbb{P}\left(\left[\frac{1}{4}, \frac{3}{4}\right]\right) + 5 \cdot \mathbb{P}([1, 10]) + 10 \cdot \mathbb{P}(\{100\}).$$

Now,

$$\mathbb{P}(A) = \sum_{k \in \mathbb{N} \cap A} \mathbb{P}(\{k\}) = \sum_{k \in \mathbb{N} \cap A} 2^{-k}.$$

Therefore

$$\int_{\Omega} X \, dP = 5 \cdot \sum_{k=1}^{10} 2^{-k} + 10 \cdot 2^{-100} = 5 + 10 \cdot 2^{-100} - 5 \cdot 2^{-10}.$$

**3.** We have that

$$\mathbb{P}(A) = \frac{1}{2} \left( \int_A \mathbf{1}_{[0,1]}(x) \, dx + \delta_{\frac{1}{2}}(A) \right),$$

where

$$\delta_{\frac{1}{2}}(A) = \begin{cases} 1 & \frac{1}{2} \in A \\ 0 & \frac{1}{2} \notin A \end{cases}.$$

Therefore,

$$\begin{aligned} \int_{\Omega} X \, dP &= \mathbb{P}\left(\left[0, \frac{1}{4}\right]\right) + 2 \cdot \mathbb{P}\left(\left\{\frac{1}{2}\right\}\right) + \mathbb{P}\left(\left[\frac{1}{2}, 1\right]\right) \\ &= \frac{1}{2} \cdot \frac{1}{4} + \frac{1}{2} \cdot 2 + \frac{1}{2} \left(\frac{1}{2} - 1 + 1\right) = \frac{11}{8}. \end{aligned}$$