

## Probability : Sheet 6 (solution)

### Problem 1

Let  $X_i$  is 1 if the  $i^{\text{th}}$  person is in favor of the new law, and 0 if the  $i^{\text{th}}$  person is not in favor the new law. Then  $X_i \sim \text{Bern}(0.65)$ . Let

$$S_{100} = X_1 + \dots + X_{100} \sim \text{Bern}(100, 0.65).$$

We denote  $\Phi$  the cumulative function of a  $\mathcal{N}(0, 1)$  law, i.e. if  $X \sim \mathcal{N}(0, 1)$ ,

$$\Phi(x) = \mathbb{P}\{X \leq x\}.$$

1. Since the  $X_i$ 's are i.i.d.<sup>1</sup>, using De Moivre Laplace theorem, we get

$$\mathbb{P}\{S_{100} \geq 50\} = \mathbb{P}\left\{\frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \geq \frac{50 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\} = 1 - \Phi\left(\frac{50 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right),$$

2. Since the  $X_i$ 's i.i.d., using De Moivre Laplace theorem, we get

$$\mathbb{P}\{S_{100} \leq 30\} = \mathbb{P}\left\{\frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{30 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\} = \Phi\left(\frac{30 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right).$$

3. Since the  $X_i$ 's i.i.d., using De Moivre Laplace theorem, we get

$$\begin{aligned} \mathbb{P}\{60 \leq S_{100} \leq 70\} &= \mathbb{P}\left\{\frac{60 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{70 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\} \\ &= \Phi\left(\frac{70 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right) - \Phi\left(\frac{60 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right). \end{aligned}$$

### Problem 2

Let  $X_i$  be 1 if the dice show 6 at the  $i^{\text{th}}$  thrown and 0 if it show something else at the  $i^{\text{th}}$  thrown. Then  $X_i \sim \text{Bern}\left(\frac{1}{6}\right)$ . Let

$$S_{1000} = X_1 + \dots + X_{1000} \sim \text{Binom}\left(1000, \frac{1}{6}\right).$$

1. Using a Poisson approximation,

$$\mathbb{P}\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda},$$

where  $\lambda = \frac{1000}{6}$ . Therefore,

$$\mathbb{P}\{150 \leq X \leq 200\} = \sum_{k=150}^{200} \mathbb{P}\{X = k\} = e^{-\lambda} \sum_{k=150}^{200} \frac{\lambda^k}{k!}.$$

We unfortunately can't write this sum as a closed form. One can use a calculator to compute the sum.

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1. independent and identically distributed

2. We have that

$$\mathbb{P} \left\{ \frac{150 - 1000 \cdot \frac{1}{6}}{\sqrt{1000 \cdot \frac{5}{36}}} \leq \frac{S_{1000} - 1000 \cdot \frac{1}{6}}{\sqrt{1000 \cdot \frac{5}{36}}} \leq \frac{200 - 1000 \cdot \frac{1}{6}}{\sqrt{1000 \cdot \frac{5}{36}}} \right\},$$

and since the  $X_i$ 's i.i.d., we get using De Moivre-Laplace,

$$\mathbb{P}\{150 \leq S_{1000} \leq 200\} = \Phi \left( \frac{200 - 1000 \cdot \frac{1}{6}}{\sqrt{1000 \cdot \frac{5}{36}}} \right) - \Phi \left( \frac{150 - 1000 \cdot \frac{1}{6}}{\sqrt{1000 \cdot \frac{5}{36}}} \right).$$

### Problem 3

Let  $X_i$  the price of a stock at the  $i^{th}$  period. We have that

$$\mathbb{P}\{X_{i+1} = uX_i\} = p \quad \text{and} \quad \mathbb{P}\{X_{i+1} = dX_i\} = 1 - p.$$

Let  $X_0 = s$ . We have to compute

$$\mathbb{P}\{X_{1000} \geq 1.3s\}.$$

Set  $Y_i = X_i - X_{i-1}$  for  $i = 1, \dots, 1000$ . We have that the  $Y_i$ 's are independents (by hypothesis),  $Y_i \sim \text{Bern}(0.52)$  for all  $i$  and

$$X_{1000} = s + Y_1 + \dots + Y_{1000}.$$

Set

$$S_{1000} = Y_1 + \dots + Y_{1000}.$$

Therefore,

$$\begin{aligned} \mathbb{P}\{X_{1000} \geq 1.3s\} &= \mathbb{P}\{S_{1000} \geq 0.3s\} = \mathbb{P} \left\{ \frac{S_{1000} - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}} \geq \frac{0.3s - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}} \right\} \\ &= 1 - \Phi \left( \frac{0.3s - 1000 \cdot 0.52}{\sqrt{1000 \cdot 0.52 \cdot 0.48}} \right). \end{aligned}$$

As we can remark, this probability depend on  $s$ , and thus, if  $s$  is to big, this probability will be 0, and if  $s$  is very small, then this will happen with probability close to 1.

### Problem 5

1. The space is  $(\Omega, \mathcal{F}, \mathbb{P})$  where  $\Omega$  is the set of the people considered,  $\mathcal{F} = 2^\Omega$  is the power set of  $\Omega$  and  $\mathbb{P}\{\omega\} = \frac{1}{10^5}$ .
2. Let  $X$  describe the number of people that have the disease. Then,

$$\mathbb{P}\{X \geq 4\} = 1 - \mathbb{P}\{X \leq 3\} = 1 - \sum_{k=0}^3 \binom{5 \cdot 10^5}{k} \frac{1}{10^{5k}} \left(1 - \frac{1}{10^5}\right)^{5 \cdot 10^5 - k} = \dots$$

3. In this situation ( $p$  very small and  $n$  very large), the Poisson approximation is a very good approximation. We set  $\lambda = \frac{4}{10^5}$ . Then,

$$\mathbb{P}\{X \geq 4\} = 1 - \mathbb{P}\{X \leq 3\} = 1 - e^{-\lambda} \sum_{k=0}^3 \frac{\lambda^k}{k!} = \dots$$

which is much much ... much easier to compute.  
 $n$  times

### Problem 7

1. It's not since  $\{1, \dots, 5\}^c = \{6\} \notin \mathcal{F}_1$ .
2. Yes it is. One can check that  $\mathcal{F}_2$  is stable by finite union, finite intersection and complementary (it's a bit long, but quite easy).
3. No it's not since  $\{1, 2, 3, \} \cap \{1, 2\}^c = \{3\} \notin \mathcal{F}_3$ .
4. No it's not since  $\{1, 2\} \cap \{2, 3, 4, 5, 6\} = \{2\} \notin \mathcal{F}_4$ .

### Problem 8

1. For all  $A \in \mathcal{F}$ ,

$$\mathbb{P}(A | B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

2. The event  $A, B \in \mathcal{F}$  are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

### Problem 9

We'll prove only that

$$\mathcal{B}(\mathbb{R}) = \sigma\{(a, b) \mid -\infty < a < b < +\infty\}$$

where  $\sigma(\mathcal{A})$  is the  $\sigma$ -algebra generated by  $\mathcal{A}$  (i.e. the smallest  $\sigma$ -algebra that contain  $\mathcal{A}$ ). The others are proved in the same way.

Let  $a, b \in \mathbb{R}$  s.t.  $a < b$ . Remark that

$$(a, b) = \bigcup_{n=1}^{\infty} \left( a, b + \frac{1}{n} \right).$$

Since  $I_n := (a, b + \frac{1}{n})$  are open for all  $n$ ,  $I_n$  belongs to  $\mathcal{B}(\mathbb{R})$  for all  $n$ . Since  $\mathcal{B}(\mathbb{R})$  is stable by countable union, we get that  $(a, b) \in \mathcal{B}(\mathbb{R})$ . Therefore

$$\sigma\{(a, b) \mid -\infty < a < b < +\infty\} \subset \mathcal{B}(\mathbb{R}).$$

Therefore  $\mathcal{B}(\mathbb{R})$  contains  $\{(a, b) \mid -\infty < a < b < +\infty\}$  and thus it contains  $\sigma\{(a, b) \mid -\infty < a < b < +\infty\}$ . Therefore

$$\sigma\{(a, b) \mid -\infty < a < b < +\infty\} \subset \mathcal{B}(\mathbb{R}).$$

We recall that  $\mathcal{B}(\mathbb{R})$  is the smallest  $\sigma$ -algebra that contains all opens sets. We therefore have to show that  $\sigma\{(a, b) \mid -\infty < a < b < +\infty\}$  contains the open set to conclude that

$$\mathcal{B}(\mathbb{R}) \subset \sigma\{(a, b) \mid -\infty < a < b < +\infty\}.$$

All open set are countable union of open interval of the form  $(a, b)$  with  $a, b \in \mathbb{R}$ . So if we prove that  $(a, b) \in \sigma\{(a, b) \mid -\infty < a < b < +\infty\}$  for all  $a, b \in \mathbb{R}$  we are done. Let  $N$  s.t.  $b - \frac{1}{n} > a$  for all  $n \geq N$ . Therefore

$$(a, b) = \bigcup_{n \geq N} \left( a, b - \frac{1}{n} \right].$$

Since  $\sigma\{(a, b) \mid -\infty < a < b < +\infty\}$  is stable by countable union and that  $(a, b - \frac{1}{n}] \in \sigma\{(a, b) \mid -\infty < a < b < +\infty\}$  for all  $n \geq N$ , we get that  $(a, b) \in \sigma\{(a, b) \mid -\infty < a < b < +\infty\}$ , and thus  $\sigma\{(a, b) \mid -\infty < a < b < +\infty\}$  contain the opens set of  $\mathbb{R}$  and thus

$$\mathcal{B}(\mathbb{R}) \subset \sigma\{(a, b) \mid -\infty < a < b < +\infty\}.$$

Finally we have that

$$\mathcal{B}(\mathbb{R}) = \sigma\{(a, b) \mid -\infty < a < b < +\infty\}.$$

### Problem 10

We have that

$$\{x\} = \bigcup_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n}\right).$$

Since  $I_n := (x - \frac{1}{n}, x + \frac{1}{n})$  is open for all  $n$ , all  $I_n$  belong to  $\mathcal{B}(\mathbb{R})$ . Since  $\mathcal{B}(\mathbb{R})$  is stable by countable union,  $\bigcup_{n \in \mathbb{N}} I_n \in \mathcal{B}(\mathbb{R})$ . Therefore  $\{x\} \in \mathcal{B}(\mathbb{R})$ .