Prof. Dr. M. Hinz
Christo Schreier

Bielefeld University

Probability : Sheet 6 (solution)

Problem 1

Let X_i is 1 if the i^{th} person is in favor of the new law, and 0 if the i^{th} person is not in favor the new law. Then $X_i \sim Bern(0.65)$. Let

$$S_{100} = X_1 + \ldots + X_{100} \sim Bern(100, 0.65).$$

We denote Φ the cumulative function of a $\mathcal{N}(0,1)$ law, i.e. if $X \sim \mathcal{N}(0,1)$,

$$\Phi(x) = \mathbb{P}\{X \leq x\}.$$

1. Since the X_i's are i.i.d. [1], using De Moivre Laplace theorem, we get

$$\mathbb{P}\{S_{100} \geq 50\} = \mathbb{P}\left\{\frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \geq \frac{50 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\} = 1 - \Phi\left(\frac{50 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right),$$

2. Since the X_i's i.i.d., using De Moivre Laplace theorem, we get

$$\mathbb{P}\{S_{100} \leq 30\} = \mathbb{P}\left\{\frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{30 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\} = \Phi\left(\frac{30 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right).$$

3. Since the X_i's i.i.d., using De Moivre Laplace theorem, we get

$$\mathbb{P}\{60 \leq S_{100} \leq 70\} = \mathbb{P}\left\{\frac{60 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{S_{100} - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}} \leq \frac{70 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right\}$$

$$= \Phi\left(\frac{70 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right) - \Phi\left(\frac{60 - 100 \cdot 0.65}{\sqrt{100 \cdot 0.65 \cdot 0.35}}\right).$$

Problem 2

Let X_i be 1 if the dice show 6 at the i^{th} thrown and 0 if it show something else at the i^{th} thrown. Then $X_i \sim Bern\left(\frac{1}{6}\right)$. Let

$$S_{1000} = X_1 + \ldots + X_{1000} \sim Binom\left(1000, \frac{1}{6}\right).$$

1. Using a Poisson approximation,

$$\mathbb{P}\{X = k\} = \frac{\lambda^k}{k!} e^{-\lambda},$$

where $\lambda = \frac{1000}{6}$. Therefore,

$$\mathbb{P}\{150 \leq X \leq 200\} = \sum_{k=150}^{200} \mathbb{P}\{X = k\} = e^{-\lambda} \sum_{k=150}^{200} \frac{\lambda^k}{k!}.$$

We unfortunately can't write this sum as a closed form. One can use a calculator to compute the sum.

1. independent and identically distributed

2. We have that

$$\mathbb{P}\left\{\frac{150-1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{5}{36}}}\le\frac{S_{1000}-1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{5}{36}}}\le\frac{200-1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{5}{36}}}\right\},$$

and since the X_i's i.i.d., we get using De Moivre-Laplace,

$$\mathbb{P}\{150\le S_{1000}\le 200\}=\Phi\left(\frac{200-1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{5}{36}}}\right)-\Phi\left(\frac{150-1000\cdot\frac{1}{6}}{\sqrt{1000\cdot\frac{5}{36}}}\right).$$

Problem 3

Let X_i the price of a stock at the i^{th} period. We have that

$$\mathbb{P}\{X_{i+1}=uX_i\}=p \quad \text{and} \quad \mathbb{P}\{X_{i+1}=dX_i\}=1-p.$$

Let $X_0=s$. We have to compute

$$\mathbb{P}\left\{X_{1000}\ge 1.3s\right\}.$$

Set $Y_i=X_i-X_{i-1}$ for $i=1,\dots,1000$. We have that the Y_i's are independents (by hypothesis), $Y_i\sim Bern(0.52)$ for all i and

$$X_{1000}=s+Y_1+\ldots+Y_{1000}.$$

Set

$$S_{1000}=Y_1+\ldots+Y_{1000}.$$

Therefore,

$$\mathbb{P}\{X_{1000}\ge 1.3s\}=\mathbb{P}\{S_{1000}\ge 0.3s\}=\mathbb{P}\left\{\frac{S_{1000}-1000\cdot 0.52}{\sqrt{1000\cdot 0.52\cdot 0.48}}\ge\frac{0.3s-1000\cdot 0.52}{\sqrt{1000\cdot 0.52\cdot 0.48}}\right\}$$
$$=1-\Phi\left(\frac{0.3s-1000\cdot 0.52}{\sqrt{1000\cdot 0.52\cdot 0.48}}\right).$$

As we can remark, this probability depend on s, and thus, if s is to big, this probability will be 0, and if s is very small, then this will happen with probability close to 1.

Problem 5

1. The space is $(\Omega,\mathcal{F},\mathbb{P})$ where Ω is the set of the people considered, $\mathcal{F}=2^\Omega$ is the power set of Ω and $\mathbb{P}\{\omega\}=\frac{1}{10^5}$.

2. Let X describe the number of people that have the disease. Then,

$$\mathbb{P}\{X\ge 4\}=1-\mathbb{P}\{X\le 3\}=1-\sum_{k=0}^{3}\binom{5\cdot 10^5}{k}\frac{1}{10^{5k}}\left(1-\frac{1}{10^5}\right)^{5\cdot 10^5-k}=...$$

3. In this situation (p very small and n very large), the Poisson approximation is a very good approxiamation. We set $\lambda=\frac{4}{10^5}$. Then,

$$\mathbb{P}\{X\ge 4\}=1-\mathbb{P}\{X\le 3\}=1-e^{-\lambda}\sum_{k=0}^{3}\frac{\lambda^k}{k!}=...$$

which is $\underbrace{\text{much much ... much}}_{n \text{ times}}$ easier to compute.

Problem 7

1. It's not since $\{1,\dots,5\}^c = \{6\} \notin \mathcal{F}_1$.

2. Yes it is. One can check that $\mathcal{F}_2$ is stable by finite union, finite intersection and complementary (it's a bit long, but quite easy).

3. No it's not since $\{1,2,3,\} \cap \{1,2\}^c = \{3\} \notin \mathcal{F}_3$.

4. No it's not since $\{1,2\} \cap \{2,3,4,5,6\} = \{2\} \notin \mathcal{F}_4$.

Problem 8

1. For all $A \in \mathcal{F}$,

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}.$$

2. The event $A, B \in \mathcal{F}$ are independent if

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B).$$

Problem 9

We'll prove only that

$$\mathcal{B}(\mathbb{R}) = \sigma\{(a,b] \mid -\infty < a < b < +\infty\}$$

where $\sigma(\mathcal{A})$ is the $\sigma-$algebra generated by $\mathcal{A}$ (i.e. the smallest $\sigma-$algebra that contain $\mathcal{A}$). The others are proved in the same way.

Let $a, b \in \mathbb{R}$ s.t. $a < b$. Remark that

$$(a,b] = \bigcup_{n=1}^{\infty} \left(a, b + \frac{1}{n}\right).$$

Since $I_n := \left(a, b + \frac{1}{n}\right)$ are open for all n, I_n belongs to $\mathcal{B}(\mathbb{R})$ for all n. Since $\mathcal{B}(\mathbb{R})$ is stable by countable union, we get that $[a,b) \in \mathcal{B}(\mathbb{R})$. Therefore

$$\sigma\{(a,b] \mid -\infty < a < b < +\infty\} \subset \mathcal{B}(\mathbb{R}).$$

Therefore $\mathcal{B}(\mathbb{R})$ contains $\{(a,b] \mid -\infty < a < b < +\infty\}$ and thus it contains $\sigma\{(a,b] \mid -\infty < a < b < +\infty\}$. Therefore

$$\{(a,b] \mid -\infty < a < b < +\infty\} \subset \mathcal{B}(\mathbb{R}).$$

We recall that $\mathcal{B}(\mathbb{R})$ is the smallest $\sigma-$algebra that contains all opens sets. We therefore have to show that $\sigma\{(a,b] \mid -\infty < a < b < +\infty\}$ contains the open set to conclude that

$$\mathcal{B}(\mathbb{R}) \subset \sigma\{(a,b] \mid -\infty < a < b < +\infty\}.$$

All open set are countable union of open interval of the form (a,b) with $a, b \in \mathbb{R}$. So if we prove that $(a,b) \in \sigma\{(a,b] \mid -\infty < a < b < +\infty\}$ for all $a, b \in \mathbb{R}$ we are done. Let N s.t. $b - \frac{1}{n} > a$ for all $n \geq N$. Therefore

$$(a,b) = \bigcup_{n \geq N} \left(a, b - \frac{1}{n}\right].$$

Since $\sigma\{(a,b] \mid -\infty < a < b < +\infty\}$ is stable by countable union and that $\left(a, b - \frac{1}{n}\right] \in \sigma\{(a,b] \mid -\infty < a < b < +\infty\}$ for all $n \geq N$, we get that $(a,b) \in \sigma\{(a,b] \mid -\infty < a < b < +\infty\}$, and thus $\sigma\{(a,b] \mid -\infty < a < b < +\infty\}$ contain the opens set of $\mathbb{R}$ and thus

$$\mathcal{B}(\mathbb{R}) \subset \sigma\{(a,b] \mid -\infty < a < b < +\infty\}.$$

Finally we have that

$$\mathcal{B}(\mathbb{R}) = \sigma\{(a,b] \mid -\infty < a < b < +\infty\}.$$

Problem 10

We have that

$$\{x\} = \bigcup_{n=1}^{\infty} \left(x - \frac{1}{n}, x + \frac{1}{n} \right).$$

Since $I_n := \left(x - \frac{1}{n}, x + \frac{1}{n}\right)$ is open for all n, all I_n belong to $\mathcal{B}(\mathbb{R})$. Since $\mathcal{B}(\mathbb{R})$ is stable by countable union, $\bigcup_{n \in \mathbb{N}} I_n \in \mathcal{B}(\mathbb{R})$. Therefore $\{x\} \in \mathcal{B}(\mathbb{R})$.