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Probability : Sheet 5 (solution)

Problem 2

We just have to prove that $\mathbb{P}(\Omega) = 1$, and all properties of a measure will follow. Set

$$\mathcal{A}_k = \big\{(\omega_1, \dots, \omega_n) \in \Omega \mid |\{\omega_i = 1 \mid i = 1, \dots, n\}| = k\big\}.$$ [1]

Then $\Omega = \bigcup_{k=0}^n \mathcal{A}_i$ and the union is disjoint. Remark that il $\omega \in \mathcal{A}_k$, then $\sum_{i=1}^n \omega_i = k$, and thus

$$\mathbb{P}\{\omega\} = p^k (1-p)^{n-k}.$$

Moreover, $|\mathcal{A}_k| = \binom{n}{k}$. Therefore

$$\mathbb{P}(\Omega_k) = \binom{n}{k} p^k (1-p)^{n-k},$$

and thus,

$$\mathbb{P}(\Omega) = \sum_{k=0}^n \binom{n}{k} p^k (1-p)^{n-k} = 1,$$

where the last inequality come from binomial theorem.

Problem 3

If S_n is a $\mathcal{B}_{n,p}$ r.v., then

$$\mathbb{P}\{S_n = k\} = \binom{n}{k} p^k (1-p)^{n-k}.$$

Suppose $n \geq 2$. Therefore,

$$\mathbb{E}S_n = \sum_{k=0}^n k\mathbb{P}\{S_n = k\} = \sum_{k=0}^n k \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} = \sum_{k=1}^n \underbrace{\frac{n!}{(n-k)!(k-1)!}}_{=n\binom{n-1}{k-1}} p^k (1-p)^{n-k}$$

$$= np \sum_{k=1}^n \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k} = np \underbrace{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-1-k}}_{=1} = np.$$

For the variance, we know that

$$\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2.$$

1. For example, if we take $n = 3$, then $\mathcal{A}_0 = \{(0,0,0)\}$, $\mathcal{A}_1 = \{(1,0,0),(0,1,0),(0,0,1)\}$, $\mathcal{A}_2 = \{(1,1,0),(0,1,1),(1,0,1)\}$ and $\mathcal{A}_3 = \{(1,1,1)\}$.

Moreover,

$$\begin{aligned}
\mathbb{E}[S_n^2] &= \sum_{k=0}^{n} k^2 \mathbb{P}\{S_n = k\} = \sum_{k=0}^{n} k^2 \frac{n!}{(n-k)!k!} p^k (1-p)^{n-k} = \sum_{k=1}^{n} k \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k} \\
&= \sum_{k=1}^{n} (k-1) \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k} + \underbrace{\sum_{k=1}^{n} \frac{n!}{(n-k)!(k-1)!} p^k (1-p)^{n-k}}_{=\mathbb{E}[S_n]} \\
&= n(n-1) \sum_{k=2}^{n} \frac{n-2}{(n-k)!(k-2)!} p^k (1-p)^{n-k} + np = n(n-1)p^2 \sum_{k=2}^{n} \binom{n-2}{k-2} p^{k-2} (1-p)^{n-k} + np \\
&= n(n-1)p^2 \underbrace{\sum_{k=0}^{n-2} \binom{n-2}{k} p^k (1-p)^{n-2-k}}_{=1} + np = n(n-1)p^2 + np.
\end{aligned}$$

Therefore

$$\mathrm{Var}(S_n) = \mathbb{E}[S_n^2] - \mathbb{E}[S_n]^2 = n(n-1)p^2 + np - n^2p^2 = np(1-p).$$

Problem 4

Let X_i the r.v. that is 1 if the i^{th} letter is incorrect and 0 otherwise. So X_i are independent and follow a Bernoulli distribution with parameter $1-p$. We have that $S := S_n = X_1 + \ldots + X_n$, that is the sum of n independent Bernoulli of parameter $1-p$, and thus S follow a Binomial distribution $\mathcal{B}(n, 1-p)$. Therefore

$$\mathbb{P}\{S = r\} = \binom{n}{r} (1-p)^r p^{n-r}.$$

Problem 5

Let X_i the random variable that is 1 if the i^{th} children go to school. The X_i are independent and follow a Bernoulli distribution of parameter 0.62. Let $S_{500} = X_1 + \ldots + X_{500}$. It's a sum of 500 independent Bernoulli r.v. of parameter 0.62, and thus S_{100} follow a Binomial distribution $\mathcal{B}(500, 0.62)$. Therefore

$$\mathbb{P}\{S_{500} \geq 290\} = \sum_{k=290}^{500} \binom{500}{k} 0.62^k 0.38^{500-k}.$$

Problem 6

1. Left to the reader.

2. Left to the reader.

3. Let X a random variable that follow a $Geom_p$ distribution. We have to prove that $\mathbb{P}\{X \in \mathbb{N}^*\} = 1$.

$$\mathbb{P}\{X \in \mathbb{N}^*\} = \sum_{k=1}^{\infty} \mathbb{P}\{X = k\} = \sum_{k=1}^{\infty} (1-p)^{k-1} p$$

$$= p \sum_{k=0}^{\infty} (1-p)^k = p \cdot \frac{1}{1-(1-p)} = p \cdot \frac{1}{p} = 1.$$

4.

$$\mathbb{E}[Y] = \sum_{k=1}^{\infty} k \mathbb{P}\{X = k\} = p \sum_{k=1}^{\infty} k(1-p)^k = p \sum_{k=1}^{\infty} k(1-p)^{k-1}$$

The sum

$$\sum_{k=1}^{\infty} kq^{k-1} = \frac{\mathrm{d}}{\mathrm{d}x}\bigg|_{x=q} \frac{1}{1-x} = \frac{1}{(1-x)^2},$$

and thus

$$\sum_{k=1}^{\infty} k(1-p)^{k+1} = \frac{1}{(1-(1-p))^2} = \frac{1}{p^2}.$$

Therefore

$$\mathbb{E}[Y] = p \cdot \frac{1}{p^2} = \frac{1}{p}.$$

To compute the variance, we use

$$\mathrm{Var}(Y) = \mathbb{E}[Y^2] = \mathbb{E}[Y]^2.$$

$$\mathbb{E}[Y^2] = \sum_{k=1}^{\infty} k^2 \mathbb{P}\{X = k\} = p \sum_{k=1}^{\infty} k^2 (1-p)^{k-1}.$$

We have that

$$\sum_{k=1}^{\infty} kx^k = x \sum_{k=1}^{\infty} kx^{k-1} = x \cdot \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=0}^{\infty} x^k = x \left(\frac{1}{1-x} \right)' = \frac{x}{(1-x)^2}.$$

Also

$$\sum_{k=1}^{\infty} k^2 x^{k-1} = \frac{\mathrm{d}}{\mathrm{d}x} \sum_{k=1}^{\infty} kx^k = \left(\frac{x}{(1-x)^2} \right)' = \frac{1+x}{(1-x)^3}.$$

Therefore,

$$\mathbb{E}[Y^2] = p \cdot \frac{1+(1-p)}{(1-(1-p))^3} = \frac{2-p}{p^2},$$

and thus

$$\mathrm{Var}(Y) = \frac{2-p}{p^2} - \frac{1}{p^2} = \frac{1-p}{p^2}.$$

Problem 7

1. One way it see it is the following : Let X_i, the i^{th} solar panel is defect. The X_i's are independent r.v. that follow a Bernoulli distribution of parameter 0.02. Then, $S_{100} = X_1 + \ldots + X_{100}$ follow a Binomial distribution $\mathcal{B}(100, 0.02)$. Therefore

$$\mathbb{P}\{S_{100} = 0\} = \binom{100}{0} 0.02^0 \cdot 0.98^{100} = 0.98^{100}.$$

An other way to model the problem is the following : V denote the number of the first defect panel. Then V follow a $Geom_{0.2}$ distribution. Then having no defect panel among the 100 first panel is

$$\mathbb{P}\{V > 100\} = (1-p)^{100} = 0.98^{100}.$$

2. By the memorylessness property

$$\mathbb{P}\{V \leq 1000 \mid V > 100\} = 1 - \mathbb{P}\{V > 1000 \mid V > 100\} = 1 - \mathbb{P}\{V > 900\} = 1-(1-0.2)^{900} = 1-0.98^{900}.$$

3. Using problem 6, we have that

$$\mathbb{E}[V] = \frac{1}{0.02} = 50 \quad \text{and} \quad \mathrm{Var}(V) = 2450.$$

Problem 8

We recall that $X \sim Poiss(\lambda)$ if

$$\mathbb{P}\{X = k\} = \frac{\lambda^k}{k!}e^{-\lambda}, \quad k \in \mathbb{N}.$$

$$\mathbb{E}[X] = \sum_{k=0}^{\infty} k\mathbb{P}\{X = k\} = \sum_{k=0}^{\infty} k\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)^2} = e^{-\lambda}\lambda\sum_{k=0}^{\infty}\frac{\lambda^k}{k!} = e^{-\lambda}\lambda e^{\lambda} = \lambda.$$

We also have

$$\mathbb{E}[X^2] = \sum_{k=0}^{\infty} k^2\frac{\lambda^k}{k!}e^{-\lambda} = e^{-\lambda}\left(\sum_{k=1}^{\infty}(k-1)\frac{\lambda^k}{(k-1)!} + \sum_{k=1}^{\infty}\frac{\lambda^k}{(k-1)!}\right)$$

$$= e^{-\lambda}\left(\lambda^2\sum_{k=2}^{\infty}\frac{\lambda^{k-2}}{(k-2)!} + \lambda\sum_{k=1}^{\infty}\frac{\lambda^{k-1}}{(k-1)!}\right) = e^{-\lambda}(\lambda^2+\lambda)e^{\lambda} = \lambda^2 + \lambda.$$

Therefore,

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Facultative Exercise

Using Binomial theorem, we have

$$\left(1+\frac{x}{n}\right)^n = \sum_{k=0}^{n}\binom{n}{k}\frac{x^k}{n^k} = \sum_{k=0}^{n}\frac{x^k}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right).$$

The last equality come from the fact that

$$\begin{aligned}\frac{n!}{(n-k)!} &= n(n-1)(n-2)\cdot(n-k+1)\\ &= n^k\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\dots\left(1-\frac{k-1}{n}\right)\\ &= n^k\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right).\end{aligned}$$

Set

$$S_n = \sum_{k=0}^{n}\frac{x^k}{k!}\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right).$$

Set

$$T_n = \sum_{k=0}^{n}\frac{x^k}{k!}.$$

Fix $n \in \mathbb{N}^*$. Then, for all $k \le n$, then

$$\prod_{i=0}^{k-1}\left(1-\frac{i}{n}\right) \le 1.$$

Therefore $S_n \le T_n$ for all n, and thus

$$\sum_{k=0}^{\infty}\frac{x^k}{k!} = \lim_{n\to\infty} T_n \ge \lim_{n\to\infty} S_n = e^x.$$

For the converse inequality, let $m \le n$. Then,

$$\sum_{k=0}^{m}\frac{x^k}{k!}\prod_{i=0}^{k-1}\left(1-\frac{k}{n}\right) \le \sum_{k=0}^{n}\frac{x^k}{k!}\prod_{i=0}^{k-1}\left(1-\frac{k}{n}\right) = S_n.$$

Therefore,

$$T_m = \lim_{n\to\infty} \sum_{k=0}^{m} \frac{x^k}{k!} \prod_{i=0}^{k-1} \left(1 - \frac{k}{n}\right) \le \lim_{n\to\infty} S_n = e^x.$$

Finally, we get

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{m\to\infty} T_m \le e^x,$$

and thus

$$\sum_{k=0}^{\infty} \frac{x^k}{k!} = e^x.$$