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# **Probability :** Sheet 4 (solution)

## Problem 1

Define the random variable $Y : \Omega \to \mathbb{R}$ by $Y(\omega) = \omega$. In particular, $\mathbb{P}\{Y \in \Omega\} = 1$.

**1.** If we set $g(k) = 2^k$, $k \in \mathbb{N}^*$, then $X = g(Y)$. By a formulae of the lecture

$$\mathbb{E}[X] = \mathbb{E}[g(Y)] = \sum_{k=1} g(k)\mathbb{P}\{Y = k\} = \sum_{k=1}^{\infty} 2^k \mathbb{P}\{Y = k\} = \sum_{k=1}^{\infty} 2^k \cdot 2^{-k} = \infty.$$

**2.** If we set $f(k) = \left(\frac{3}{2}\right)^k$, $k \in \mathbb{N}^*$ then $Y = g(Y)$. By a formulae of the lecture

$$\mathbb{E}[X] = \mathbb{E}[g(Y)] = \sum_{k=1}^{\infty} f(k)\mathbb{P}\{Y = k\} = \sum_{k=1}^{\infty} \left(\frac{3}{4}\right)^k = 3.$$

**3.** If we set $h(k) = (-1)^k 2^k$, $k \in \mathbb{N}^*$, then $X = h(Y)$ and thus

$$\mathbb{E}[X] = \lim_{n\to\infty} \sum_{k=1}^{n} g(k)\mathbb{P}\{Y = k\} = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^k 2^k \cdot \frac{1}{2^k} = \lim_{n\to\infty} \sum_{k=1}^{n} (-1)^k,$$

and since the last limit doesn't exist, the expectation doesn't exist.

## Problem 3

Let $Y(\omega) = \omega$. Then the range of $Y$ is $\Omega$, and thus, since $X = |Y|$, we get

$$\mathbb{E}[X] = \mathbb{E}[|Y|] = \sum_{k=-2}^{2} |k|^2 \mathbb{P}\{Y = k\} = 2 \cdot 2 \cdot \frac{1}{8} + 2 \cdot \frac{1}{8} + 0 \cdot \frac{1}{2} = \frac{3}{4}.$$

To compute the variance, we use the fact that

$$\mathrm{Var}(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

Since $X^2 = Y^2$, we have

$$\mathbb{E}[X^2] = \sum_{k=-2}^{2} k^2 \mathbb{P}\{Y = k\} = 2 \cdot 4 \cdot \frac{1}{8} + 2 \cdot \frac{1}{8} + 0 \cdot \frac{1}{2} = \frac{5}{4}.$$

Therefore

$$\mathrm{Var}(X) = \frac{5}{4} - \frac{9}{16} = \frac{11}{16}.$$

## Problem 5

We have that $\mathbb{P}\{X = k\} = \frac{1}{n}$ for all $k$. Therefore

$$\mathbb{E}[X] = \sum_{k=1}^{n} k\mathbb{P}\{X = k\} = \frac{1}{n} \sum_{k=1}^{n} k = \frac{1}{n} \cdot \frac{n(n+1)}{2} = \frac{n+1}{2}.$$

Let $Y = X^2$. The range of $Y$ is $\mathcal{D} = \{k^2 \mid k = 1, \dots, n\}$. Therefore,

$$\begin{aligned}
\mathbb{E}[Y] &= \sum_{j \in \mathcal{D}} j\mathbb{P}\{Y = j\} = \sum_{k=1}^{n} k^2 \mathbb{P}\{X = k\} = \frac{1}{n}\sum_{k=1}^{n} k^2 \\
&= \frac{1}{n} \cdot \frac{n(n+1)(2n+1)}{6} = \frac{(n+1)(2n+1)}{6} \\
&= \frac{(n+1)}{2} \cdot \frac{(2n+1)}{3}.
\end{aligned}$$

Since $\frac{2n+1}{3} \geq \frac{n+1}{2}$ for all $n \geq 1$, (you can prove it by induction), we get

$$\mathbb{E}[X]^2 = \frac{n+1}{2} \cdot \frac{n+1}{2} \leq \frac{n+1}{2} \cdot \frac{2n+1}{3} = \mathbb{E}[X^2].$$

## Problem 6

By definition

$$\operatorname{Var}(X) = \mathbb{E}\big[(X - \mathbb{E}[X])^2\big].$$

**1.** Using the exercise 8 of the sheet 3, we know that $X \leq Y$ a.s. implies $\mathbb{E}[X] \leq \mathbb{E}[Y]$. Since

$$(X - \mathbb{E}[X])^2 \geq 0 \quad \text{a.s.},$$

we get $\operatorname{Var}(X) \geq 0$.

**2.** Using linearity of expectation,

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b,$$

and thus

$$\mathbb{E}\left[\Big(aX + b - \mathbb{E}[aX + b]\Big)^2\right] = a^2\mathbb{E}\left[\Big(X - \mathbb{E}[X]\Big)^2\right] = a^2\operatorname{Var}(X).$$

**3.** Since

$$\big(X - \mathbb{E}[X]\big)^2 = X^2 - 2x\mathbb{E}[X] + \mathbb{E}[X]^2,$$

we get using linearity of the expectation (and recalling that $\mathbb{E}[X] \in \mathbb{R}$) that

$$\operatorname{Var}(X) = \mathbb{E}\Big[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2\Big] = \mathbb{E}[X^2] - 2\mathbb{E}[X]\mathbb{E}[X] + \mathbb{E}[X]^2 = \mathbb{E}[X^2] - \mathbb{E}[X]^2.$$

## Problem 7

Using Exercise 8 of sheet 3, we know that $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A)$. Therefore, by **3.** from the previous exercise,

$$\operatorname{Var}(\mathbf{1}_A) = \mathbb{E}[\mathbf{1}_A^2] - \mathbb{E}[\mathbf{1}_A]^2 = \mathbb{P}(A) - \mathbb{P}(A)^2 = \mathbb{P}(A)\big(1 - \mathbb{P}(A)\big) = \mathbb{P}(A)\mathbb{P}(A^c).$$

## Problem 8

For me, Chebychev inequality and Markov inequality are the same... but in the litterature they can be a bit different. The idea is anyway to remark that

$$|X| \geq |X|\mathbf{1}_{\{|X| \geq a\}} \geq a\mathbf{1}_{\{|X| \geq a\}},$$

i.e.

$$|X| \geq a\mathbf{1}_{\{|X| \geq a\}}.$$

Using the fact that $\mathbb{E}[\mathbf{1}_A] = \mathbb{P}(A)$ (see exercise 8 of the sheet 3) and the linearity of the expectation, we get

$$\mathbb{E}\big[|X|\big] \geq a\mathbb{P}\left\{\big|X\big| \geq a\right\},$$

or more commonly

$$\mathbb{P}\{|X| \geq a\} \leq \frac{\mathbb{E}\big[|X|\big]}{a}.$$

Using this, you can get

$$\mathbb{P}\left\{\left|X-\mathbb{E}[X]\right|>\varepsilon\right\}=\mathbb{P}\left\{\left|X-\mathbb{E}[X]\right|^2>\varepsilon^2\right\}\leq\frac{\mathrm{Var}(X)}{\varepsilon^2},$$

it's maybe what you call Chebychev inequality ?