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Probability : Sheet 3 (solution)

Problem 1

There are two ways to see the problem. I'll present both. I denote R the event *"the result is red"* and G the event *"the result is green"*.

1. For $RGRRR$ we can see it as 5 independent Bernoulli experiment with parameter $\frac{2}{5}$ (if we consider that the success is R). So, if R_i is "having a red at the i^{th} thrown" and $G_i = R_i^c$, then

$$\mathbb{P}(RGRRR) = \mathbb{P}(R_1 \cap G_2 \cap R_3 \cap R_4 \cap R_5) = \mathbb{P}(R_1) \cdot \mathbb{P}(G_2) \cdot \mathbb{P}(R_3) \cdot \mathbb{P}(R_4) \cdot \mathbb{P}(R_5) = \frac{2^4 \cdot 5}{6^5} \approx 8,3 \cdot 10^{-3}.$$

For $RGRRRG$ and $GRRRRR$ it correspond to 6 independent Bernoulli experiment with parameter $\frac{2}{6}$ (if the success is still R). By a similar calculation, we get respectively for $RGRRRG$ and $GRRRRR$

$$\frac{2^4 \cdot 4^2}{6^6} \approx 5,4 \cdot 10^{-3} \quad \text{and} \quad \frac{4 \cdot 2^5}{6^6} \approx 2,7 \cdot 10^{-3}.$$

2. The second way is to consider the sample space

$$\Omega = \{(i_1, \dots, i_5) \mid i_1, \dots, i_5 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\}$$

with

$$\mathbb{P}\{(i_1, \dots, i_5)\} = \frac{1}{|\Omega|} = \frac{1}{6^5}$$

for $RGRRR$ and

$$\tilde{\Omega}\{(i_1, \dots, i_6) \mid i_1, \dots, i_6 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\},$$

with

$$\mathbb{P}\{(i_1, \dots, i_6)\} = \frac{1}{|\tilde{\Omega}|} = \frac{1}{6^6}$$

for $RGRRRG$ and $GRRRRR$.

To get $RGRRR$, is associated to the event

$$E = \{(i_1, \dots, i_5) \mid i_2 \in \{G_1, \dots, G_4\}, i_1, i_3, i_4, i_5 \in \{R_1, R_2\}\},$$

and thus

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{4 \cdot 2^4}{6^5}.$$

For $RGRRRG$ and $GRRRRR$, the events are respectively

$$F = \{(i_1, \dots, i_6) \mid i_1, i_3, i_4, i_5 \in \{R_1, R_2\}, i_2, i_6 \in \{G_1, G_2\}\},$$

and

$$G = \{(i_1, \dots, i_6) \mid i_1 \in \{G_1, \dots, G_4\}, i_2, i_3, i_4, i_5, i_6 \in \{R_1, R_2\}\}.$$

Therefore,

$$\mathbb{P}(F) = \frac{|F|}{|\tilde{\Omega}|} = \frac{2^4 \cdot 4^2}{6^6} \quad \text{and} \quad \mathbb{P}(G) = \frac{|G|}{|\tilde{\Omega}|} = \frac{4 \cdot 2^5}{6^6}.$$

Problem 2

1. We have to write

$$X+Y=\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}+\sum_{k=1}^{\infty} y_k \mathbf{1}_{\{Y=y_k\}},$$

as a sum of the form

$$\sum_{k,\ell} c_{k,\ell}\mathbf{1}_{A_{k,\ell}}.$$

Notice that

$$\mathbf{1}_{\Omega}=\mathbf{1}_{\bigcup_{i=1}^{\infty}\{X=x_k\}}=\sum_{i=1}^{\infty}\mathbf{1}_{\{X=x_k\}}$$

and

$$\mathbf{1}_{\Omega}=\mathbf{1}_{\bigcup_{i=1}^{\infty}\{Y=y_k\}}=\sum_{i=1}^{\infty}\mathbf{1}_{\{Y=y_k\}},$$

where in both cases, the last inequality come from exercise 5 of this sheet exercise. We have that

$$\begin{aligned}\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} &= \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\Omega}=\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\bigcup_{i=1}^{\infty}\{Y=y_i\}}\\ &=\sum_{k=1}^{\infty}\sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\{Y=y_i\}} \underset{(*)}{=} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k,Y=y_i\}},\end{aligned}$$

where $(*)$ come from the exercise 5 of this sheet. In the same way, we get

$$\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{Y=y_j\}}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

One can easily prove that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}},$$

and thus

$$X+Y=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(x_k+y_j)\mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

2. The result is immediate using exercise 5 of this sheet, and we get

$$XY=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} x_k y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

3. Let $\omega\in\Omega$. Then $\omega\in\{X=x_k\}$ for some k. In particular

$$g(X)(\omega):=g(X(\omega))=g(x_k).$$

Therefore,

$$g(X)(\omega)=\sum_{k=1}^{\infty} g(x_k)\mathbf{1}_{\{X=x_k\}}.$$

Problem 5

We have to prove that for all $\omega \in \Omega$,

$$\mathbf{1}_{A\cap B}(\omega) = \mathbf{1}_A(\omega)\mathbf{1}_B(\omega)$$

and if A and B are disjoints,

$$\mathbf{1}_{A\cup B}(\omega) = \mathbf{1}_A(\omega) + \mathbf{1}_B(\omega)$$

1. We have by definition

$$\mathbf{1}_{A\cap B}(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A\cap B)^c \end{cases}.$$

Moreover

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = 1 \iff \mathbf{1}_A(\omega) = 1 \text{ and } \mathbf{1}_B(\omega) = 1 \iff \omega \in A \text{ and } \omega \in B \iff \omega \in A \cap B,$$

and

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = 0 \iff \mathbf{1}_A(\omega) = 0 \text{ or } \mathbf{1}_B(\omega) = 0 \iff \omega \notin A \text{ or } \omega \notin B$$
$$\iff x \in A^c \text{ or } x \in B^c \iff \omega \in (A^c \cup B^c) = (A\cap B)^c.$$

Therefore,

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A\cap B)^c \end{cases} = \mathbf{1}_{A\cap B}(\omega).$$

2. The proof essentially goes the same. Here, the fact that A and B are disjoints gives

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff (\omega \in A \text{ and } \omega \notin B) \text{ or } (\omega \in B \text{ and } \omega \notin A) \iff \omega \in (A\cap B^c)\cup(B\cap A^c).$$

Since A and B are disjoint, $A \cap B^c = A$ and $B \cap A^c = B$. Therefore,

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff \omega \in A \cup B.$$

The rest of the proof is left to the reader.

Problem 6

This is typically the kind of proof that is very easy to write but very hard to follow. I add an example below to show you what happen. When you understood the example and the mechanism behind, you should be able to do it by your self (and you'll see how easy it is). But if you go into it without previous training, good luck ! ;-)

Let

$$X = \sum_{k=1}^{n} x_k \mathbf{1}_{A_k}.$$

Suppose with out loss of generality that $x_1, \ldots, x_k$ are all distincts ($k \leq n$) and k maximal in the sense that if $\ell > k$, then $x_\ell = x_j$ for some $j \in \{1, \ldots, k\}$. For all $i \in \{1, \ldots, n\}$, set

$$J_i = \{j \in \{1, \ldots, n\} \mid x_j = x_i\}.$$

Notice that if $\ell \in J_i$ then $J_\ell = J_i$. Remark that if $x_i \neq x_j$, then J_i and J_j are disjoints. Indeed, let $i \neq j$ with $x_i \neq x_j$. If $x \in J_i \cap J_j$, then $x = x_i = x_j$ which contradict $x_i \neq x_j$. Therefore $J_i \cap J_j = \varnothing$. Finally, by maximality on k,

$$\bigcup_{i=1}^{k} J_i = \{1, \ldots, n\},$$

and the union is disjoints. Set

$$A_{J_p} = \bigcup_{j \in J_p} A_j. \tag{1}$$

Then for all p,

$$X(\omega) = x_p \iff \omega \in A_{J_p}.$$

By Problem 5, if A and B are disjoints, then $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$, and thus, we can write X as

$$X = \sum_{i=1}^{k} x_i \mathbf{1}_{A_{J_i}} \underset{(1)}{=} \sum_{i=1}^{k} x_i \mathbf{1}_{\{X=x_i\}}.$$

By definition of the expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^{k} x_i \mathbb{P}\{X = x_i\} = \sum_{i=1}^{k} x_i \mathbb{P}(A_{J_i}) \\ &\underset{(a)}{=} \sum_{i=1}^{k} x_i \sum_{j\in J_i} \mathbb{P}(A_j) = \sum_{i=1}^{k} \sum_{j\in J_i} x_i \mathbb{P}(A_j) \\ &\underset{(b)}{=} \sum_{i=1}^{k} \sum_{j\in J_i} x_j \mathbb{P}(A_j) = \sum_{j\in J_1\cup\ldots\cup J_k} x_j \mathbb{P}(A_j) \\ &= \sum_{j=1}^{n} x_j \mathbb{P}(A_j),\end{aligned}$$

where (a) is because the A_i's are disjoints, and (b) because if $j \in J_i$ then $x_i = x_j$. The claim is proved.

Example

Here an example on how the proof works. Let consider consider $\Omega = \{1, \ldots, 6\}$ and the function

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega).$$

I can also write it as

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega).$$

Now, let write $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 2$, $x_5 = 2$ and $x_6 = 3$ and $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{6\}$, $A_4 = \{3\}$, $A_5 = \{4\}$ and $A_6 = \{5\}$. So the A_i's are disjoints, and

$$X(\omega) = \sum_{i=1}^{6} x_i \mathbf{1}_{A_i}(\omega).$$

The k in the beginning of the proof correspond here to 3. So the list $x_1, \ldots, x_k$ of distinct elements in the proof is x_1, x_2, x_3 here and correspond to $2, 3, 5$. As you can see, adding any x_i for $i > 3$ in x_1, x_2, x_3 will add an element that is already in x_1, x_2, x_3. For example, if I want to add $x_5 = 2$, then 2 will appear twice (since $x_1 = 2$). So, the list x_1, x_2, x_3 is maximal in the sense that adding any element of x_4, x_5, x_6 will add an element that already exist in x_1, x_2, x_3. Now,

$$J_1 = \big\{i \in \{1, \ldots, 6\} \mid x_i = x_1\big\} = \big\{i \in \{1, \ldots, 6\} \mid x_i = 2\big\} = \{1, 3, 4\},$$

since $x_1 = x_4 = x_5 = 2$. Same,

$$J_2 = \big\{i \in \{1, \ldots, 6\} \mid x_i = x_2\big\} = \big\{i \in \{1, \ldots, 6\} \mid x_i = 3\big\} = \{2, 5\},$$

since $x_2 = x_5 = 3$. Finally,

$$J_3 = \big\{i \in \{1, \ldots, 6\} \mid x_i = x_3\big\} = \big\{i \in \{2, \ldots, 6\} \mid x_i = 5\big\} = \{3\}.$$

As you can remark, the J_i's are disjoints and

$$J_1 \cup J_2 \cup J_3 = \{1, \ldots, 6\}. \tag{2}$$

Now we set,

$$A_{J_1} = A_1 \cup A_3 \cup A_4, \quad A_{J_2} = A_2 \cup A_5, \quad \text{and} \quad A_{J_3} = A_3.$$

We can then write X as the sum

$$\begin{aligned} X(\omega) &= 2 \cdot (\mathbf{1}_{\{1\}}(\omega) + \mathbf{1}_{\{3\}}(\omega) + \mathbf{1}_{\{4\}}(\omega)) + 3 \cdot (\mathbf{1}_{\{2\}}(\omega) + \mathbf{1}_{\{5\}}(\omega)) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{\{1,3,4\}}(\omega) + 3 \cdot \mathbf{1}_{\{2,5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot 1_{A_{J_1}} + 3 \cdot \mathbf{1}_{A_{J_2}} + 5 \cdot \mathbf{1}_{A_{J_5}}, \end{aligned}$$

and since

$$A_{J_1} = \{X = 2\}, \quad A_{J_2} = \{X = 3\} \quad \text{and} \quad A_{J_3} = \{X = 5\}, \tag{3}$$

we finally get

$$X(\omega) = 0 \cdot \mathbf{1}_{\{X=1\}}(\omega) + 2 \cdot \mathbf{1}_{\{X=2\}}(\omega) + 3 \cdot \mathbf{1}_{\{X=3\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=4\}}(\omega) + 5 \cdot \mathbf{1}_{\{X=5\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=6\}}(\omega).$$

By definition of the expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= 0 \cdot \mathbb{P}\{X = 1\} + 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 0 \cdot \mathbb{P}\{X = 4\} + 5 \cdot \mathbb{P}\{X = 5\} + 0 \cdot \mathbb{P}\{X = 6\} \\ &= 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 5 \cdot \mathbb{P}\{X = 5\} \\ &\underset{(3)}{=} 2 \cdot \mathbb{P}(A_{J_1}) + 3 \cdot \mathbb{P}(A_{J_2}) + 5 \cdot \mathbb{P}(A_{J_3}). \end{aligned}$$

Now, since the A_i's are disjoints, we have that

$$\mathbb{P}(A_{J_1}) = \mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \quad \text{and} \quad \mathbb{P}(A_{J_2}) = \mathbb{P}(A_2) + \mathbb{P}(A_5),$$

so

$$\begin{aligned} \mathbb{E}[X] &= 2 \cdot \big(\mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4)\big) + 3 \cdot \big(\mathbb{P}(A_2) + \mathbb{P}(A_5)\big) + 5 \cdot \mathbb{P}(A_6) \\ &= \big(2 \cdot \mathbb{P}(A_1) + 2 \cdot \mathbb{P}(A_4) + 2 \cdot \mathbb{P}(A_5)\big) + \big(3 \cdot \mathbb{P}(A_2) + 3 \cdot \mathbb{P}(A_5)\big) + 5 \cdot \mathbb{P}(A_6) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} 2\mathbb{P}(A_j) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) \end{aligned}$$

But remember that if $j \in J_i$, then $x_j = x_i$. For example, for $j \in J_1$, we have

$$\sum_{j \in J_1} x_1 \mathbb{P}(A_j) = \sum_{j \in \{1,3,4\}} x_1 \mathbb{P}(A_j) = x_1 \mathbb{P}(A_1) + x_1 \mathbb{P}(A_3) + x_1 \mathbb{P}(A_4). \tag{4}$$

But since $x_1 = x_3 = x_4 = 2$, the right hand side of (4) is nothing more than

$$x_1 \mathbb{P}(A_1) + x_3 \mathbb{P}(A_3) + x_4 \mathbb{P}(A_4) = \sum_{j \in \{1,3,4\}} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j).$$

So finally,

$$\sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) = \sum_{i=1}^{3} \sum_{j \in J_i} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j) + \sum_{j \in J_2} x_j \mathbb{P}(A_j) + \sum_{j \in J_3} x_j \mathbb{P}(A_j) \tag{5}$$

and since the J_i's are disjoints, the right hand side of (5) is

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j).$$

But as (2), $J_1 \cup J_2 \cup J_3 = \{1, \ldots, 6\}$ and thus

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j) = \sum_{j=1}^{6} x_j \mathbb{P}(A_j).$$

Finally, we conclude that $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{i=1}^{6} x_i \mathbb{P}(A_i),$$

as wished.

Problem 7

The prove goes by induction. For $n = 1$, the result is obviously correct. I recall that a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$, and all $\lambda \in [0, 1]$,

$$g\big(\lambda x + (1 - \lambda)y\big) \leq \lambda g(x) + (1 - \lambda)g(y).$$

Suppose that

$$g\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i g(x_i),$$

for $\displaystyle\sum_{i=1}^{n} a_i = 1$. Let $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ s.t. $\sum_{i=1}^{n+1} \lambda_i = 1$. Remark that

$$\sum_{i=1}^{n+1} \lambda_i x_i = (1 - \lambda_{n+1}) \underbrace{\frac{\lambda_1 x_1 + \ldots + \lambda_n x_n}{1 - \lambda_{n+1}}}_{=:y} + \lambda_{n+1} x_{n+1} = (1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}.$$

Then

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = g\big((1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}\big) \underset{(*)}{\leq} (1 - \lambda_{n+1})g(y) + \lambda_{n+1} g(x_{n+1}), \tag{6}$$

where $(*)$ come from the definition of the convexity. If we set $a_i = \frac{\lambda_i}{1 - \lambda_{n+1}}$, then

$$y = a_1 x_1 + \ldots + a_n x_n$$

and

$$\sum_{i=1}^{n} a_i = \frac{\lambda_1 + \ldots + \lambda_n}{1 - \lambda_{n+1}} = \frac{1 - \lambda_{n+1}}{1 - \lambda_{n+1}} = 1.$$

Therefore, by recurrence hypothesis,

$$g(y) \leq \sum_{i=1}^{n} g(x_i) a_i,$$

and thus

$$(1 - \lambda_{n+1})g(y) \leq (1 - \lambda_{n+1}) \sum_{i=1}^{n} a_i g(x_i) = \sum_{i=1}^{n} \lambda_i g(x_i). \tag{7}$$

Combine (6) and (7) gives

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq \sum_{i=1}^{n+1} \lambda_i g(x_i).$$