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Probability : Sheet 3 (solution)

Problem 1

There are two ways to see the problem. I'll present both. I denote R the event *"the result is red"* and G the event *"the result is green"*.

1. For $RGRRR$ we can see it as 5 independent Bernoulli experiment with parameter $\frac{2}{5}$ (if we consider that the success is R). So, if R_i is "having a red at the i^{th} thrown" and $G_i = R_i^c$, then

$$\mathbb{P}(RGRRR) = \mathbb{P}(R_1 \cap G_2 \cap R_3 \cap R_4 \cap R_5) = \mathbb{P}(R_1) \cdot \mathbb{P}(G_2) \cdot \mathbb{P}(R_3) \cdot \mathbb{P}(R_4) \cdot \mathbb{P}(R_5) = \frac{2^4 \cdot 5}{6^5} \approx 8,3 \cdot 10^{-3}.$$

For $RGRRRG$ and $GRRRRR$ it correspond to 6 independent Bernoulli experiment with parameter $\frac{2}{6}$ (if the success is still R). By a similar calculation, we get respectively for $RGRRRG$ and $GRRRRR$

$$\frac{2^4 \cdot 4^2}{6^6} \approx 5,4 \cdot 10^{-3} \quad \text{and} \quad \frac{4 \cdot 2^5}{6^6} \approx 2,7 \cdot 10^{-3}.$$

2. The second way is to consider the sample space

$$\Omega = \{(i_1, \dots, i_5) \mid i_1, \dots, i_5 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\}$$

with

$$\mathbb{P}\{(i_1, \dots, i_5)\} = \frac{1}{|\Omega|} = \frac{1}{6^5}$$

for $RGRRR$ and

$$\tilde{\Omega}\{(i_1, \dots, i_6) \mid i_1, \dots, i_6 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\},$$

with

$$\mathbb{P}\{(i_1, \dots, i_6)\} = \frac{1}{|\tilde{\Omega}|} = \frac{1}{6^6}$$

for $RGRRRG$ and $GRRRRR$.

To get $RGRRR$, is associated to the event

$$E = \{(i_1, \dots, i_5) \mid i_2 \in \{G_1, \dots, G_4\}, i_1, i_3, i_4, i_5 \in \{R_1, R_2\}\},$$

and thus

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{4 \cdot 2^4}{6^5}.$$

For $RGRRRG$ and $GRRRRR$, the events are respectively

$$F = \{(i_1, \dots, i_6) \mid i_1, i_3, i_4, i_5 \in \{R_1, R_2\}, i_2, i_6 \in \{G_1, G_2\}\},$$

and

$$G = \{(i_1, \dots, i_6) \mid i_1 \in \{G_1, \dots, G_4\}, i_2, i_3, i_4, i_5, i_6 \in \{R_1, R_2\}\}.$$

Therefore,

$$\mathbb{P}(F) = \frac{|F|}{|\tilde{\Omega}|} = \frac{2^4 \cdot 4^2}{6^6} \quad \text{and} \quad \mathbb{P}(G) = \frac{|G|}{|\tilde{\Omega}|} = \frac{4 \cdot 2^5}{6^6}.$$

Problem 2

1. We have to write

$$X+Y=\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}+\sum_{k=1}^{\infty} y_k \mathbf{1}_{\{Y=y_k\}},$$

as a sum of the form

$$\sum_{k,\ell} c_{k,\ell}\mathbf{1}_{A_{k,\ell}}.$$

Notice that

$$\mathbf{1}_{\Omega}=\mathbf{1}_{\bigcup_{i=1}^{\infty}\{X=x_k\}}=\sum_{i=1}^{\infty}\mathbf{1}_{\{X=x_k\}}$$

and

$$\mathbf{1}_{\Omega}=\mathbf{1}_{\bigcup_{i=1}^{\infty}\{Y=y_k\}}=\sum_{i=1}^{\infty}\mathbf{1}_{\{Y=y_k\}},$$

where in both cases, the last inequality come from exercise 5 of this sheet exercise. We have that

$$\begin{aligned}\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} &= \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\Omega} = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\bigcup_{i=1}^{\infty}\{Y=y_i\}} \\ &= \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}}\mathbf{1}_{\{Y=y_i\}} \underset{(*)}{=} \sum_{k=1}^{\infty}\sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k,Y=y_i\}},\end{aligned}$$

where $(*)$ come from the exercise 5 of this sheet. In the same way, we get

$$\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{Y=y_j\}}=\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

One can easily prove that

$$\sum_{j=1}^{\infty}\sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}},$$

and thus

$$X+Y=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty}(x_k+y_j)\mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

2. The result is immediate using exercise 5 of this sheet, and we get

$$XY=\sum_{k=1}^{\infty}\sum_{j=1}^{\infty} x_k y_j \mathbf{1}_{\{X=x_k,Y=y_j\}}.$$

3. Let $\omega\in\Omega$. Then $\omega\in\{X=x_k\}$ for some k. In particular

$$g(X)(\omega):=g(X(\omega))=g(x_k).$$

Therefore,

$$g(X)(\omega)=\sum_{k=1}^{\infty} g(x_k)\mathbf{1}_{\{X=x_k\}}.$$

Problem 5

We have to prove that for all $\omega \in \Omega$,

$$\mathbf{1}_{A\cap B}(\omega) = \mathbf{1}_A(\omega)\mathbf{1}_B(\omega)$$

and if A and B are disjoints,

$$\mathbf{1}_{A\cup B}(\omega) = \mathbf{1}_A(\omega) + \mathbf{1}_B(\omega)$$

1. We have by definition

$$\mathbf{1}_{A\cap B}(\omega) = \begin{cases} 1 & \omega \in A\cap B \\ 0 & \omega \in (A\cap B)^c \end{cases}.$$

Moreover

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = 1 \iff \mathbf{1}_A(\omega) = 1 \text{ and } \mathbf{1}_B(\omega) = 1 \iff \omega \in A \text{ and } \omega \in B \iff \omega \in A\cap B,$$

and

$$\begin{aligned} \mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = 0 &\iff \mathbf{1}_A(\omega) = 0 \text{ or } \mathbf{1}_B(\omega) = 0 \iff \omega \notin A \text{ or } \omega \notin B \\ &\iff x \in A^c \text{ or } x \in B^c \iff \omega \in (A^c \cup B^c) = (A\cap B)^c. \end{aligned}$$

Therefore,

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega) = \begin{cases} 1 & \omega \in A\cap B \\ 0 & \omega \in (A\cap B)^c \end{cases} = \mathbf{1}_{A\cap B}(\omega).$$

2. The proof essentially goes the same. Here, the fact that A and B are disjoints gives

$$\mathbf{1}_A(\omega)+\mathbf{1}_B(\omega) = 1 \iff (\omega \in A \text{ and } \omega \notin B) \text{ or } (\omega \in B \text{ and } \omega \notin A) \iff \omega \in (A\cap B^c)\cup(B\cap A^c).$$

Since A and B are disjoint, $A\cap B^c = A$ and $B \cap A^c = B$. Therefore,

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff \omega \in A\cup B.$$

The rest of the proof is left to the reader.

Problem 6

Let

$$X = \sum_{k=1}^{n} x_k \mathbf{1}_{A_k}.$$

Suppose with out loss of generality that $x_1, \ldots, x_k$ are all distincts ($k \leq n$) and k maximal in the sense that if $\ell > k$, then $x_\ell = x_j$ for some $j \in \{1, \ldots, k\}$. For all $i \in \{1, \ldots, k\}$, set

$$J_i = \{j \in \{1, \ldots, n\} \mid x_j = x_i\}.$$

Notice that if $\ell \in J_i$ then $J_\ell = J_i$. Remark that if $x_i \neq x_j$, then J_i and J_j are disjoints. Indeed, let $i \neq j$ with $x_i \neq x_j$. If $x \in J_i \cap J_j$, then $x = x_i = x_j$ which contradict $x_i \neq x_j$. Therefore $J_i \cap J_j = \varnothing$. Finally, by assumption on k and maximality of k,

$$\bigcup_{i=1}^{k} J_i = \{1, \ldots, n\},$$

and the union is disjoints. Set

$$A_{J_i} = \bigcup_{j\in J_i} A_j. \tag{1}$$

Then, for all i,

$$X(\omega) = x_i \iff \omega \in A_{J_i}.$$

By Problem 5, if A and B are disjoints, then $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$, and thus, we can write X as

$$X = \sum_{i=1}^{k} x_i \mathbf{1}_{A_{J_i}} \underset{(1)}{=} \sum_{i=1}^{k} x_i \mathbf{1}_{\{X=x_i\}}.$$

By definition of the expectation,

$$\begin{aligned}\mathbb{E}[X] &= \sum_{i=1}^{k} x_i \mathbb{P}\{X = x_i\} = \sum_{i=1}^{k} x_i \mathbb{P}(A_{J_i}) \\ &\underset{(a)}{=} \sum_{i=1}^{k} x_i \sum_{j\in J_i} \mathbb{P}(A_j) = \sum_{i=1}^{k} \sum_{j\in J_i} x_i \mathbb{P}(A_j) \\ &\underset{(b)}{=} \sum_{i=1}^{k} \sum_{j\in J_i} x_j \mathbb{P}(A_j) = \sum_{j\in J_1\cup\ldots\cup J_k} x_j \mathbb{P}(A_j) \\ &= \sum_{j=1}^{n} x_j \mathbb{P}(A_j),\end{aligned}$$

where (a) is because the A_i's are disjoints, and (b) because if $j \in J_i$ then $x_i = x_j$. The claim is proved.

Example

Here an example on how the proof works. Let consider consider $\Omega = \{1, \ldots, 6\}$ and the function

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega).$$

I can also write it as

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega).$$

Now, let write $x_1 = 2$, $x_2 = 3$, $x_3 = 5$, $x_4 = 2$, $x_5 = 2$ and $x_6 = 3$ and $A_1 = \{1\}$, $A_2 = \{2\}$, $A_3 = \{6\}$, $A_4 = \{3\}$, $A_5 = \{4\}$ and $A_6 = \{5\}$. So the A_i's are disjoints, and

$$X(\omega) = \sum_{i=1}^{6} x_i \mathbf{1}_{A_i}(\omega).$$

The k in the beginning of the proof correspond here to 3. So the list $x_1, \ldots, x_k$ of distinct elements in the proof is x_1, x_2, x_3 here and correspond to $2, 3, 5$. As you can see, adding any x_i for $i > 3$ in x_1, x_2, x_3 will add an element that is already in x_1, x_2, x_3. For example, if I want to add $x_5 = 2$, then 2 will appear twice (since $x_1 = 2$). So, the list x_1, x_2, x_3 is maximal in the sense that adding any element of x_4, x_5, x_6 will add an element that already exist in x_1, x_2, x_3. Now,

$$J_1 = \Big\{i \in \{1, \ldots, 6\} \mid x_i = x_1\Big\} = \Big\{i \in \{1, \ldots, 6\} \mid x_i = 2\Big\} = \{1, 3, 4\},$$

since $x_1 = x_4 = x_5 = 2$. Same,

$$J_2 = \Big\{i \in \{1, \ldots, 6\} \mid x_i = x_2\Big\} = \Big\{i \in \{1, \ldots, 6\} \mid x_i = 3\Big\} = \{2, 5\},$$

since $x_2 = x_5 = 3$. Finally,

$$J_3 = \Big\{i \in \{1, \ldots, 6\} \mid x_i = x_3\Big\} = \Big\{i \in \{2, \ldots, 6\} \mid x_i = 5\Big\} = \{3\}.$$

As you can remark, the J_i's are disjoints and

$$J_1 \cup J_2 \cup J_3 = \{1, \ldots, 6\}. \tag{2}$$

Now we set,

$$A_{J_1} = A_1 \cup A_3 \cup A_4, \quad A_{J_2} = A_2 \cup A_5, \quad \text{and} \quad A_{J_3} = A_3.$$

We can then write X as the sum

$$\begin{aligned} X(\omega) &= 2 \cdot (\mathbf{1}_{\{1\}}(\omega) + \mathbf{1}_{\{3\}}(\omega) + \mathbf{1}_{\{4\}}(\omega)) + 3 \cdot (\mathbf{1}_{\{2\}}(\omega) + \mathbf{1}_{\{5\}}(\omega)) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{\{1,3,4\}}(\omega) + 3 \cdot \mathbf{1}_{\{2,5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot 1_{A_{J_1}} + 3 \cdot \mathbf{1}_{A_{J_2}} + 5 \cdot \mathbf{1}_{A_{J_5}}, \end{aligned}$$

and since

$$A_{J_1} = \{X = 2\}, \quad A_{J_2} = \{X = 3\} \quad \text{and} \quad A_{J_3} = \{X = 5\}, \tag{3}$$

we finally get

$$X(\omega) = 0 \cdot \mathbf{1}_{\{X=1\}}(\omega) + 2 \cdot \mathbf{1}_{\{X=2\}}(\omega) + 3 \cdot \mathbf{1}_{\{X=3\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=4\}}(\omega) + 5 \cdot \mathbf{1}_{\{X=5\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=6\}}(\omega).$$

By definition of the expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= 0 \cdot \mathbb{P}\{X = 1\} + 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 0 \cdot \mathbb{P}\{X = 4\} + 5 \cdot \mathbb{P}\{X = 5\} + 0 \cdot \mathbb{P}\{X = 6\} \\ &= 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 5 \cdot \mathbb{P}\{X = 5\} \\ &\underset{(3)}{=} 2 \cdot \mathbb{P}(A_{J_1}) + 3 \cdot \mathbb{P}(A_{J_2}) + 5 \cdot \mathbb{P}(A_{J_3}). \end{aligned}$$

Now, since the A_i's are disjoints, we have that

$$\mathbb{P}(A_{J_1}) = \mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \quad \text{and} \quad \mathbb{P}(A_{J_2}) = \mathbb{P}(A_2) + \mathbb{P}(A_5),$$

so

$$\begin{aligned} \mathbb{E}[X] &= 2 \cdot \big(\mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4)\big) + 3 \cdot \big(\mathbb{P}(A_2) + \mathbb{P}(A_5)\big) + 5 \cdot \mathbb{P}(A_6) \\ &= \big(2 \cdot \mathbb{P}(A_1) + 2 \cdot \mathbb{P}(A_4) + 2 \cdot \mathbb{P}(A_5)\big) + \big(3 \cdot \mathbb{P}(A_2) + 3 \cdot \mathbb{P}(A_5)\big) + 5 \cdot \mathbb{P}(A_6) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} 2\mathbb{P}(A_j) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) \end{aligned}$$

But remember that if $j \in J_i$, then $x_j = x_i$. For example, for $j \in J_1$, we have

$$\sum_{j \in J_1} x_1 \mathbb{P}(A_j) = \sum_{j \in \{1,3,4\}} x_1 \mathbb{P}(A_j) = x_1 \mathbb{P}(A_1) + x_1 \mathbb{P}(A_3) + x_1 \mathbb{P}(A_4). \tag{4}$$

But since $x_1 = x_3 = x_4 = 2$, the right hand side of (4) is nothing more than

$$x_1 \mathbb{P}(A_1) + x_3 \mathbb{P}(A_3) + x_4 \mathbb{P}(A_4) = \sum_{j \in \{1,3,4\}} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j).$$

So finally,

$$\sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) = \sum_{i=1}^{3} \sum_{j \in J_i} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j) + \sum_{j \in J_2} x_j \mathbb{P}(A_j) + \sum_{j \in J_3} x_j \mathbb{P}(A_j) \tag{5}$$

and since the J_i's are disjoints, the right hand side of (5) is

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j).$$

But as (2), $J_1 \cup J_2 \cup J_3 = \{1, \ldots, 6\}$ and thus

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j) = \sum_{j=1}^{6} x_j \mathbb{P}(A_j).$$

Finally, we conclude that $\mathbb{E}[X]$ is

$$\mathbb{E}[X] = \sum_{i=1}^{6} x_i \mathbb{P}(A_i),$$

as wished.

Problem 7

The prove goes by induction. For $n = 1$, the result is obviously correct. I recall that a function $g : \mathbb{R} \longrightarrow \mathbb{R}$ is convex if for all $x, y \in \mathbb{R}$, and all $\lambda \in [0, 1]$,

$$g\big(\lambda x + (1 - \lambda)y\big) \leq \lambda g(x) + (1 - \lambda)g(y).$$

Suppose that

$$g\left(\sum_{i=1}^{n} a_i x_i\right) \leq \sum_{i=1}^{n} a_i g(x_i),$$

for $\displaystyle\sum_{i=1}^{n} a_i = 1$. Let $\lambda_1, \ldots, \lambda_{n+1} \geq 0$ s.t. $\sum_{i=1}^{n+1} \lambda_i = 1$. Remark that

$$\sum_{i=1}^{n+1} \lambda_i x_i = (1 - \lambda_{n+1}) \underbrace{\frac{\lambda_1 x_1 + \ldots + \lambda_n x_n}{1 - \lambda_{n+1}}}_{=:y} + \lambda_{n+1} x_{n+1} = (1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}.$$

Then

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = g\big((1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}\big) \underset{(*)}{\leq} (1 - \lambda_{n+1})g(y) + \lambda_{n+1} g(x_{n+1}), \tag{6}$$

where $(*)$ come from the definition of the convexity. If we set $a_i = \frac{\lambda_i}{1 - \lambda_{n+1}}$, then

$$y = a_1 x_1 + \ldots + a_n x_n$$

and

$$\sum_{i=1}^{n} a_i = \frac{\lambda_1 + \ldots + \lambda_n}{1 - \lambda_{n+1}} = \frac{1 - \lambda_{n+1}}{1 - \lambda_{n+1}} = 1.$$

Therefore, by recurrence hypothesis,

$$g(y) \leq \sum_{i=1}^{n} g(x_i) a_i,$$

and thus

$$(1 - \lambda_{n+1}) g(y) \leq (1 - \lambda_{n+1}) \sum_{i=1}^{n} a_i g(x_i) = \sum_{i=1}^{n} \lambda_i g(x_i). \tag{7}$$

Combine (6) and (7) gives

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq \sum_{i=1}^{n+1} \lambda_i g(x_i).$$