

## Probability : Sheet 3 (solution)

### Problem 1

There are two ways to see the problem. I'll present both. I denote  $R$  the event "the result is red" and  $G$  the event "the result is green".

1. For  $RGRRR$  we can see it as 5 independent Bernoulli experiment with parameter  $\frac{2}{5}$  (if we consider that the success is  $R$ ). So, if  $R_i$  is "having a red at the  $i^{\text{th}}$  thrown" and  $G_i = R_i^c$ , then

$$\mathbb{P}(RGRRR) = \mathbb{P}(R_1 \cap G_2 \cap R_3 \cap R_4 \cap R_5) = \mathbb{P}(R_1) \cdot \mathbb{P}(G_2) \cdot \mathbb{P}(R_3) \cdot \mathbb{P}(R_4) \cdot \mathbb{P}(R_5) = \frac{2^4 \cdot 5}{6^5} \approx 8,3 \cdot 10^{-3}.$$

For  $RGRRRG$  and  $GRRRRR$  it correspond to 6 independent Bernoulli experiment with parameter  $\frac{2}{6}$  (if the success is still  $R$ ). By a similar calculation, we get respectively for  $RGRRRG$  and  $GRRRRR$

$$\frac{2^4 \cdot 4^2}{6^6} \approx 5,4 \cdot 10^{-3} \quad \text{and} \quad \frac{4 \cdot 2^5}{6^6} \approx 2,7 \cdot 10^{-3}.$$

2. The second way is to consider the sample space

$$\Omega = \{(i_1, \dots, i_5) \mid i_1, \dots, i_5 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\}$$

with

$$\mathbb{P}\{(i_1, \dots, i_5)\} = \frac{1}{|\Omega|} = \frac{1}{6^5}$$

for  $RGRRR$  and

$$\tilde{\Omega}\{(i_1, \dots, i_6) \mid i_1, \dots, i_6 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\},$$

with

$$\mathbb{P}\{(i_1, \dots, i_6)\} = \frac{1}{|\tilde{\Omega}|} = \frac{1}{6^6}$$

for  $RGRRRG$  and  $GRRRRR$ .

To get  $RGRRR$ , is associated to the event

$$E = \{(i_1, \dots, i_5) \mid i_2 \in \{G_1, \dots, G_4\}, i_1, i_3, i_4, i_5 \in \{R_1, R_2\}\},$$

and thus

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{4 \cdot 2^4}{6^5}.$$

For  $RGRRRG$  and  $GRRRRR$ , the events are respectively

$$F = \{(i_1, \dots, i_6) \mid i_1, i_3, i_4, i_5 \in \{R_1, R_2\}, i_2, i_6 \in \{G_1, G_2\}\},$$

and

$$G = \{(i_1, \dots, i_6) \mid i_1 \in \{G_1, \dots, G_4\}, i_2, i_3, i_4, i_5, i_6 \in \{R_1, R_2\}\}.$$

Therefore,

$$\mathbb{P}(F) = \frac{|F|}{|\tilde{\Omega}|} = \frac{2^4 \cdot 4^2}{6^6} \quad \text{and} \quad \mathbb{P}(G) = \frac{|G|}{|\tilde{\Omega}|} = \frac{4 \cdot 2^5}{6^6}.$$

## Problem 2

1. We have to write

$$X + Y = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} + \sum_{k=1}^{\infty} y_k \mathbf{1}_{\{Y=y_k\}},$$

as a sum of the form

$$\sum_{k,\ell} c_{k,\ell} \mathbf{1}_{A_{k,\ell}}.$$

Notice that

$$\mathbf{1}_{\Omega} = \mathbf{1}_{\bigcup_{i=1}^{\infty} \{X=x_k\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{X=x_k\}}$$

and

$$\mathbf{1}_{\Omega} = \mathbf{1}_{\bigcup_{i=1}^{\infty} \{Y=y_k\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{Y=y_k\}},$$

where in both cases, the last inequality come from exercise 5 of this sheet exercise.

We have that

$$\begin{aligned} \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} &= \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\Omega} = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\bigcup_{i=1}^{\infty} \{Y=y_i\}} \\ &= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\{Y=y_i\}} \stackrel{(*)}{=} \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k, Y=y_i\}}, \end{aligned}$$

where (\*) come from the exercise 5 of this sheet. In the same way, we get

$$\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{Y=y_j\}} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k, Y=y_j\}}.$$

One can easily prove that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k, Y=y_j\}} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} y_j \mathbf{1}_{\{X=x_k, Y=y_j\}},$$

and thus

$$X + Y = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (x_k + y_j) \mathbf{1}_{\{X=x_k, Y=y_j\}}.$$

2. The result is immediate using exercise 5 of this sheet, and we get

$$XY = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x_k y_j \mathbf{1}_{\{X=x_k, Y=y_j\}}.$$

3. Let  $\omega \in \Omega$ . Then  $\omega \in \{X = x_k\}$  for some  $k$ . In particular

$$g(X)(\omega) := g(X(\omega)) = g(x_k).$$

Therefore,

$$g(X)(\omega) = \sum_{k=1}^{\infty} g(x_k) \mathbf{1}_{\{X=x_k\}}.$$

## Problem 5

We have to prove that for all  $\omega \in \Omega$ ,

$$\mathbf{1}_{A \cap B}(\omega) = \mathbf{1}_A(\omega) \mathbf{1}_B(\omega)$$

and if  $A$  and  $B$  are disjoint,

$$\mathbf{1}_{A \cup B}(\omega) = \mathbf{1}_A(\omega) + \mathbf{1}_B(\omega)$$

1. We have by definition

$$\mathbf{1}_{A \cap B}(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A \cap B)^c \end{cases}.$$

Moreover

$$\mathbf{1}_A(\omega) \mathbf{1}_B(\omega) = 1 \iff \mathbf{1}_A(\omega) = 1 \text{ and } \mathbf{1}_B(\omega) = 1 \iff \omega \in A \text{ and } \omega \in B \iff \omega \in A \cap B,$$

and

$$\begin{aligned} \mathbf{1}_A(\omega) \mathbf{1}_B(\omega) = 0 &\iff \mathbf{1}_A(\omega) = 0 \text{ or } \mathbf{1}_B(\omega) = 0 \iff \omega \notin A \text{ or } \omega \notin B \\ &\iff x \in A^c \text{ or } x \in B^c \iff \omega \in (A^c \cup B^c) = (A \cap B)^c. \end{aligned}$$

Therefore,

$$\mathbf{1}_A(\omega) \mathbf{1}_B(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A \cap B)^c \end{cases} = \mathbf{1}_{A \cap B}(\omega).$$

2. The proof essentially goes the same. Here, the fact that  $A$  and  $B$  are disjoint gives

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff (\omega \in A \text{ and } \omega \notin B) \text{ or } (\omega \in B \text{ and } \omega \notin A) \iff \omega \in (A \cap B^c) \cup (B \cap A^c).$$

Since  $A$  and  $B$  are disjoint,  $A \cap B^c = A$  and  $B \cap A^c = B$ . Therefore,

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff \omega \in A \cup B.$$

The rest of the proof is left to the reader.

## Problem 6

Let

$$X = \sum_{k=1}^n x_k \mathbf{1}_{A_k}.$$

Suppose with out loss of generality that  $x_1, \dots, x_k$  all distincts ( $k \geq n$ ) and  $k$  maximal in the sense that if  $\ell > k$ , then  $x_\ell = x_i$  for some  $i \in \{1, \dots, k\}$ . For all  $i \in \{1, \dots, n\}$ , set

$$J_i = \{j \in \{1, \dots, n\} \mid x_j = x_i\}.$$

Notice that if  $\ell \in J_i$  then  $J_\ell = J_i$ . Denote  $A_{J_i} = \bigcup_{j \in J_i} A_j$ . Remark that if  $x_i \neq x_j$ , then  $J_i$  and  $J_j$  are disjoint. Indeed, let  $i \neq j$  with  $x_i \neq x_j$ . If  $x \in J_i \cap J_j$ , then  $x = x_i = x_j$  which contradict  $x_i \neq x_j$ . Therefore  $J_i \cap J_j = \emptyset$ . Finally, by assumption on  $k$  and maximality of  $k$ ,

$$\bigcup_{i=1}^k J_i = \{1, \dots, n\},$$

and the union is disjoint. Set

$$A_{J_i} = \bigcup_{j \in J_i} A_j. \tag{1}$$

Then, for all  $i$ ,

$$X(\omega) = x_i \iff \omega \in A_{J_i}.$$

By Problem 5, if  $A$  and  $B$  are disjoint, then  $\mathbf{1}_{A \cup B} = \mathbf{1}_A + \mathbf{1}_B$ , and thus, we can write  $X$  as

$$X = \sum_{i=1}^k x_i \mathbf{1}_{A_{J_i}} \stackrel{(1)}{=} \sum_{i=1}^k x_i \mathbf{1}_{\{X=x_i\}}.$$

By definition of the expectation,

$$\begin{aligned} \mathbb{E}[X] &= \sum_{i=1}^k x_i \mathbb{P}\{X = x_i\} = \sum_{i=1}^k x_i \mathbb{P}(A_{J_i}) \\ &\stackrel{(a)}{=} \sum_{i=1}^k x_i \sum_{j \in J_i} \mathbb{P}(A_j) = \sum_{i=1}^k \sum_{j \in J_i} x_i \mathbb{P}(A_j) \\ &\stackrel{(b)}{=} \sum_{i=1}^k \sum_{j \in J_i} x_j \mathbb{P}(A_j) = \sum_{j \in J_1 \cup \dots \cup J_k} x_j \mathbb{P}(A_j) \\ &= \sum_{j=1}^n x_j \mathbb{P}(A_j), \end{aligned}$$

where (a) is because the  $A_i$ 's are disjoint, and (b) because if  $j \in J_i$  then  $x_i = x_j$ . The claim is proved.

### Example

Here an example on how the proof works. Let consider consider  $\Omega = \{1, \dots, 6\}$  and the function

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega).$$

I can also write it as

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega).$$

Now, let write  $x_1 = 2$ ,  $x_2 = 3$ ,  $x_3 = 5$ ,  $x_4 = 2$ ,  $x_5 = 2$  and  $x_6 = 3$  and  $A_1 = \{1\}$ ,  $A_2 = \{2\}$ ,  $A_3 = \{6\}$ ,  $A_4 = \{3\}$ ,  $A_5 = \{4\}$  and  $A_6 = \{5\}$ . So the  $A_i$ 's are disjoint, and

$$X(\omega) = \sum_{i=1}^6 x_i \mathbf{1}_{A_i}(\omega).$$

The  $k$  in the beginning of the proof correspond here to 3. So the list  $x_1, \dots, x_k$  of distinct elements in the proof is  $x_1, x_2, x_3$  here and correspond to 2, 3, 5. As you can see, adding any  $x_i$  for  $i > 3$  in  $x_1, x_2, x_3$  will add an element that is already in  $x_1, x_2, x_3$ . For example, if I want to add  $x_5 = 2$ , then 2 will appear twice (since  $x_1 = 2$ ). So, the list  $x_1, x_2, x_3$  is maximal in the sense that adding any element of  $x_4, x_5, x_6$  will add an element that already exist in  $x_1, x_2, x_3$ . Now,

$$J_1 = \{i \in \{1, \dots, 6\} \mid x_i = x_1\} = \{i \in \{1, \dots, 6\} \mid x_i = 2\} = \{1, 3, 4\},$$

since  $x_1 = x_4 = x_5 = 2$ . Same,

$$J_2 = \{i \in \{1, \dots, 6\} \mid x_i = x_2\} = \{i \in \{1, \dots, 6\} \mid x_i = 3\} = \{2, 5\},$$

since  $x_2 = x_5 = 3$ . Finally,

$$J_3 = \{i \in \{1, \dots, 6\} \mid x_i = x_3\} = \{i \in \{2, \dots, 6\} \mid x_i = 5\} = \{3\}.$$

As you can remark, the  $J_i$ 's are disjoint and

$$J_1 \cup J_2 \cup J_3 = \{1, \dots, 6\}. \tag{2}$$

Now we set,

$$A_{J_1} = A_1 \cup A_3 \cup A_4, \quad A_{J_2} = A_2 \cup A_5, \quad \text{and} \quad A_{J_3} = A_3.$$

We can then write  $X$  as the sum

$$\begin{aligned} X(\omega) &= 2 \cdot (\mathbf{1}_{\{1\}}(\omega) + \mathbf{1}_{\{3\}}(\omega) + \mathbf{1}_{\{4\}}(\omega)) + 3 \cdot (\mathbf{1}_{\{2\}}(\omega) + \mathbf{1}_{\{5\}}(\omega)) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{\{1,3,4\}}(\omega) + 3 \cdot \mathbf{1}_{\{2,5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{A_{J_1}} + 3 \cdot \mathbf{1}_{A_{J_2}} + 5 \cdot \mathbf{1}_{A_{J_3}}, \end{aligned}$$

and since

$$A_{J_1} = \{X = 2\}, \quad A_{J_2} = \{X = 3\} \quad \text{and} \quad A_{J_3} = \{X = 5\}, \quad (3)$$

we finally get

$$X(\omega) = 0 \cdot \mathbf{1}_{\{X=1\}}(\omega) + 2 \cdot \mathbf{1}_{\{X=2\}}(\omega) + 3 \cdot \mathbf{1}_{\{X=3\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=4\}}(\omega) + 5 \cdot \mathbf{1}_{\{X=5\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=6\}}(\omega).$$

By definition of the expectation, we have

$$\begin{aligned} \mathbb{E}[X] &= 0 \cdot \mathbb{P}\{X = 1\} + 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 0 \cdot \mathbb{P}\{X = 4\} + 5 \cdot \mathbb{P}\{X = 5\} + 0 \cdot \mathbb{P}\{X = 6\} \\ &= 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 5 \cdot \mathbb{P}\{X = 5\} \\ &\stackrel{(3)}{=} 2 \cdot \mathbb{P}(A_{J_1}) + 3 \cdot \mathbb{P}(A_{J_2}) + 5 \cdot \mathbb{P}(A_{J_3}). \end{aligned}$$

Now, since the  $A_i$ 's are disjoint, we have that

$$\mathbb{P}(A_{J_1}) = \mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \quad \text{and} \quad \mathbb{P}(A_{J_2}) = \mathbb{P}(A_2) + \mathbb{P}(A_5),$$

so

$$\begin{aligned} \mathbb{E}[X] &= 2 \cdot (\mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4)) + 3 \cdot (\mathbb{P}(A_2) + \mathbb{P}(A_5)) + 5 \cdot \mathbb{P}(A_6) \\ &= (2 \cdot \mathbb{P}(A_1) + 2 \cdot \mathbb{P}(A_4) + 2 \cdot \mathbb{P}(A_5)) + (3 \cdot \mathbb{P}(A_2) + 3 \cdot \mathbb{P}(A_5)) + 5 \cdot \mathbb{P}(A_6) \\ &= \sum_{i=1}^3 \sum_{j \in J_i} 2\mathbb{P}(A_j) \\ &= \sum_{i=1}^3 \sum_{j \in J_i} x_i \mathbb{P}(A_j) \end{aligned}$$

But remember that if  $j \in J_i$ , then  $x_j = x_i$ . For example, for  $j \in J_1$ , we have

$$\sum_{j \in J_1} x_1 \mathbb{P}(A_j) = \sum_{j \in \{1,3,4\}} x_1 \mathbb{P}(A_j) = x_1 \mathbb{P}(A_1) + x_1 \mathbb{P}(A_3) + x_1 \mathbb{P}(A_4). \quad (4)$$

But since  $x_1 = x_3 = x_4 = 2$ , the right hand side of (4) is nothing more than

$$x_1 \mathbb{P}(A_1) + x_3 \mathbb{P}(A_3) + x_4 \mathbb{P}(A_4) = \sum_{j \in \{1,3,4\}} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j).$$

So finally,

$$\sum_{i=1}^3 \sum_{j \in J_i} x_i \mathbb{P}(A_j) = \sum_{i=1}^3 \sum_{j \in J_i} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j) + \sum_{j \in J_2} x_j \mathbb{P}(A_j) + \sum_{j \in J_3} x_j \mathbb{P}(A_j) \quad (5)$$

and since the  $J_i$ 's are disjoint, the right hand side of (5) is

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j).$$

But as (2),  $J_1 \cup J_2 \cup J_3 = \{1, \dots, 6\}$  and thus

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j) = \sum_{j=1}^6 x_j \mathbb{P}(A_j).$$

Finally, we conclude that  $\mathbb{E}[X]$  is

$$\mathbb{E}[X] = \sum_{i=1}^6 x_i \mathbb{P}(A_i),$$

as wished.

### Problem 7

The prove goes by induction. For  $n = 1$ , the result is obviously correct. I recall that a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  is convex if for all  $x, y \in \mathbb{R}$ , and all  $\lambda \in [0, 1]$ ,

$$g(\lambda x + (1 - \lambda)y) \leq \lambda g(x) + (1 - \lambda)g(y).$$

Suppose that

$$g\left(\sum_{i=1}^n a_i x_i\right) \leq \sum_{i=1}^n a_i g(x_i),$$

for  $\sum_{i=1}^n a_i = 1$ . Let  $\lambda_1, \dots, \lambda_{n+1} \geq 0$  s.t.  $\sum_{i=1}^{n+1} \lambda_i = 1$ . Remark that

$$\sum_{i=1}^{n+1} \lambda_i x_i = (1 - \lambda_{n+1}) \underbrace{\frac{\lambda_1 x_1 + \dots + \lambda_n x_n}{1 - \lambda_{n+1}}}_{=: y} + \lambda_{n+1} x_{n+1} = (1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}.$$

Then

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = g((1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}) \stackrel{(*)}{\leq} (1 - \lambda_{n+1})g(y) + \lambda_{n+1}g(x_{n+1}), \quad (6)$$

where (\*) come from the definition of the convexity. If we set  $a_i = \frac{\lambda_i}{1 - \lambda_{n+1}}$ , then

$$y = a_1 x_1 + \dots + a_n x_n$$

and

$$\sum_{i=1}^n a_i = \frac{\lambda_1 + \dots + \lambda_n}{1 - \lambda_{n+1}} = \frac{1 - \lambda_{n+1}}{1 - \lambda_{n+1}} = 1.$$

Therefore, by recurrence hypothesis,

$$g(y) \leq \sum_{i=1}^n g(x_i) a_i,$$

and thus

$$(1 - \lambda_{n+1})g(y) \leq (1 - \lambda_{n+1}) \sum_{i=1}^n a_i g(x_i) = \sum_{i=1}^n \lambda_i g(x_i). \quad (7)$$

Combine (6) and (7) gives

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq \sum_{i=1}^{n+1} \lambda_i g(x_i).$$