# **Probability:** Sheet 3 (solution)

## Problem 1

There are two ways to see the problem. I'll present both. I denote R the event "the result is red" and G the event "the result is green".

1. For RGRRR we can see it as 5 independent Bernoulli experiment with parameter  $\frac{2}{5}$  (if we consider that the success is R). So, if  $R_i$  is "having a red at the  $i^{th}$  thrown" and  $G_i = R_i^c$ , then

$$\mathbb{P}(RGRRR) = \mathbb{P}(R_1 \cap G_2 \cap R_3 \cap R_4 \cap R_5) = \mathbb{P}(R_1) \cdot \mathbb{P}(G_2) \cdot \mathbb{P}(R_3) \cdot \mathbb{P}(R_4) \cdot \mathbb{P}(R_5) = \frac{2^4 \cdot 5}{6^5} \approx 8, 3 \cdot 10^{-3}.$$

For RGRRRG and GRRRRR it correspond to 6 independent Bernoulli experiment with parameter  $\frac{2}{6}$  (if the success is still R). By a similar calculation, we get respectively for RGRRRG and GRRRRR

$$\frac{2^4 \cdot 4^2}{6^6} \approx 5, 4 \cdot 10^{-3} \quad \text{and} \quad \frac{4 \cdot 2^5}{6^6} \approx 2, 7 \cdot 10^{-3}.$$

2. The second way is to consider the sample space

$$\Omega = \{(i_1, \dots, i_5) \mid i_1, \dots, i_5 \in \{R_1, R_2, G_1, G_2, G_3, G_4\}\}\$$

with

$$\mathbb{P}\{(i_1,\ldots,i_5)\} = rac{1}{|\Omega|} = rac{1}{6^5}$$

for RGRRR and

$$\tilde{\Omega}\{(i_1,\ldots,i_6)\mid i_1,\ldots,i_6\in\{R_1,R_2,G_1,G_2,G_3,G_4\}\},\$$

with

$$\mathbb{P}\{(i_1,\ldots,i_6)\} = \frac{1}{|\tilde{\Omega}|} = \frac{1}{6^6}$$

for RGRRRG and GRRRRR.

To get RGRRR, is associated to the event

$$E = \{(i_1, \dots, i_5) \mid i_2 \in \{G_1, \dots, G_4\}, i_1, i_3, i_4, i_5 \in \{R_1, R_2\}\},\$$

and thus

$$\mathbb{P}(E) = \frac{|E|}{|\Omega|} = \frac{4 \cdot 2^4}{6^5}.$$

For RGRRRG and GRRRRR, the events are respectively

$$F = \{(i_1, \dots, i_6) \mid i_1, i_3, i_4, i_5 \in \{R_1, R_2\}, i_2, i_6 \in \{G_1, G_2\}\},\$$

and

$$G = \{(i_1, \dots, i_6) \mid i_1 \in \{G_1, \dots, G_4\}, i_2, i_3, i_4, i_5, i_6 \in \{R_1, R_2\}\}.$$

Therefore,

$$\mathbb{P}(F) = \frac{|F|}{|\tilde{\Omega}|} = \frac{2^4 \cdot 4^2}{6^6} \quad \text{and} \quad \mathbb{P}(G) = \frac{|G|}{|\tilde{\Omega}|} = \frac{4 \cdot 2^5}{6^6}.$$

## Problem 2

1. We have to write

$$X + Y = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X = x_k\}} + \sum_{k=1}^{\infty} y_k \mathbf{1}_{\{Y = y_k\}},$$

as a sum of the form

$$\sum_{k,\ell} c_{k,\ell} \mathbf{1}_{A_{k,\ell}}.$$

Notice that

$$\mathbf{1}_{\Omega} = \mathbf{1}_{\bigcup_{i=1}^{\infty} \{X = x_k\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{X = x_k\}}$$

and

$$\mathbf{1}_{\Omega} = \mathbf{1}_{\bigcup_{i=1}^{\infty} \{Y = y_k\}} = \sum_{i=1}^{\infty} \mathbf{1}_{\{Y = y_k\}},$$

where in both cases, the last inequality come from exercise 5 of this sheet exercise. We have that

$$\sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\Omega} = \sum_{k=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\bigcup_{i=1}^{\infty} \{Y=y_i\}}$$
$$= \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k\}} \mathbf{1}_{\{Y=y_i\}} = \sum_{k=1}^{\infty} \sum_{i=1}^{\infty} x_k \mathbf{1}_{\{X=x_k,Y=y_i\}},$$

where (\*) come from the exercise 5 of this sheet. In the same way, we get

$$\sum_{j=1}^{\infty} y_j \mathbf{1}_{\{Y=y_j\}} = \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k, Y=y_j\}}.$$

One can easily prove that

$$\sum_{j=1}^{\infty} \sum_{k=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}} = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} y_j \mathbf{1}_{\{X=x_k,Y=y_j\}},$$

and thus

$$X + Y = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} (x_k + y_j) \mathbf{1}_{\{X = x_k, Y = y_j\}}.$$

2. The result is immediate using exercise 5 of this sheet, and we get

$$XY = \sum_{k=1}^{\infty} \sum_{j=1}^{\infty} x_k y_j \mathbf{1}_{\{X = x_k, Y = y_j\}}.$$

3. Let  $\omega \in \Omega$  . Then  $\omega \in \{X = x_k\}$  for some k. In particular

$$g(X)(\omega) := g(X(\omega)) = g(x_k).$$

Therefore,

$$g(X)(\omega) = \sum_{k=1}^{\infty} g(x_k) \mathbf{1}_{\{X = x_k\}}.$$

### Problem 5

We have to prove that for all  $\omega \in \Omega$ ,

$$\mathbf{1}_{A\cap B}(\omega) = \mathbf{1}_A(\omega)\mathbf{1}_B(\omega)$$

and if A and B are disjoints,

$$\mathbf{1}_{A\cup B}(\omega) = \mathbf{1}_{A}(\omega) + \mathbf{1}_{B}(\omega)$$

1. We have by definition

$$\mathbf{1}_{A\cap B}(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A \cap B)^c \end{cases}.$$

Moreover

$$\mathbf{1}_A(\omega)\mathbf{1}_B(\omega)=1\iff \mathbf{1}_A(\omega)=1 \text{ and } \mathbf{1}_B(\omega)=1 \iff \omega\in A \text{ and } \omega\in B \iff \omega\in A\cap B,$$

and

$$\mathbf{1}_{A}(\omega)\mathbf{1}_{B}(\omega) = 0 \iff \mathbf{1}_{A}(\omega) = 0 \text{ or } \mathbf{1}_{B}(\omega) = 0 \iff \omega \notin A \text{ or } \omega \notin B$$
  
 $\iff x \in A^{c} \text{ or } x \in B^{c} \iff \omega \in (A^{c} \cup B^{c}) = (A \cap B)^{c}.$ 

Therefore,

$$\mathbf{1}_{A}(\omega)\mathbf{1}_{B}(\omega) = \begin{cases} 1 & \omega \in A \cap B \\ 0 & \omega \in (A \cap B)^{c} \end{cases} = \mathbf{1}_{A \cap B}(\omega).$$

2. The proof essentially goes the same. Here, the fact that A and B are disjoints gives

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff (\omega \in A \text{ and } \omega \notin B) \text{ or } (\omega \in B \text{ and } \omega \notin A) \iff \omega \in (A \cap B^c) \cup (B \cap A^c).$$

Since A and B are disjoint,  $A \cap B^c = A$  and  $B \cap A^c = B$ . Therefore,

$$\mathbf{1}_A(\omega) + \mathbf{1}_B(\omega) = 1 \iff \omega \in A \cup B.$$

The rest of the proof is left to the reader.

### Problem 6

Let

$$X = \sum_{k=1}^{n} x_k \mathbf{1}_{A_k}.$$

Suppose with out loss of generality that  $x_1, \ldots, x_k$  all distincts  $(k \ge n)$  and k maximal in the sense that if  $\ell > k$ , then  $x_\ell = x_i$  for some  $i \in \{1, \ldots, k\}$ . For all  $i \in \{1, \ldots, n\}$ , set

$$J_i = \{j \in \{1, \dots, n\} \mid x_j = x_i\}.$$

Notice that if  $\ell \in J_i$  then  $J_\ell = J_i$ . Denote  $A_{J_i} = \bigcup_{j \in J_i} A_i$ . Remark that if  $x_i \neq x_j$ , then  $J_i$  and  $J_j$  are disjoints. Indeed, let  $i \neq j$  with  $x_i \neq x_j$ . If  $x \in J_i \cap J_j$ , then  $x = x_i = x_j$  which contradict  $x_i \neq x_j$ . Therefore  $J_i \cap J_j = \emptyset$ . Finally, by assumption on k and maximality of k,

$$\bigcup_{i=1}^k J_i = \{1, \dots, n\},\,$$

and the union is disjoints. Set

$$A_{J_i} = \bigcup_{j \in J_i} A_j. \tag{1}$$

Then, for all i,

$$X(\omega) = x_i \iff \omega \in A_{J_i}$$
.

By Problem 5, if A and B are disjoints, then  $\mathbf{1}_{A\cup B} = \mathbf{1}_A + \mathbf{1}_B$ , and thus, we can write X as

$$X = \sum_{i=1}^{k} x_i \mathbf{1}_{A_{J_i}} = \sum_{i=1}^{k} x_i \mathbf{1}_{\{X = x_i\}}.$$

By definition of the expectation,

$$\mathbb{E}[X] = \sum_{i=1}^{k} x_{i} \mathbb{P}\{X = x_{i}\} = \sum_{i=1}^{k} x_{i} \mathbb{P}(A_{J_{i}})$$

$$= \sum_{i=1}^{k} x_{i} \sum_{j \in J_{i}} \mathbb{P}(A_{j}) = \sum_{i=1}^{k} \sum_{j \in J_{i}} x_{i} \mathbb{P}(A_{j})$$

$$= \sum_{i=1}^{k} \sum_{j \in J_{i}} x_{j} \mathbb{P}(A_{j}) = \sum_{j \in J_{1} \cup ... \cup J_{k}} x_{j} \mathbb{P}(A_{j})$$

$$= \sum_{i=1}^{n} x_{j} \mathbb{P}(A_{j}),$$

where (a) is because the  $A_i$ 's are disjoints, and (b) because if  $j \in J_i$  then  $x_i = x_j$ . The claim is proved.

#### Example

Here an example on how the proof works. Let consider consider  $\Omega = \{1, \dots, 6\}$  and the function

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega).$$

I can also write it as

$$X(\omega) = 2 \cdot \mathbf{1}_{\{1\}}(\omega) + 3 \cdot \mathbf{1}_{\{2\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) + 2 \cdot \mathbf{1}_{\{3\}}(\omega) + 2 \cdot \mathbf{1}_{\{4\}}(\omega) + 3 \cdot \mathbf{1}_{\{5\}}(\omega).$$

Now, let write  $x_1 = 2$ ,  $x_2 = 3$ ,  $x_3 = 5$ ,  $x_4 = 2$ ,  $x_5 = 2$  and  $x_6 = 3$  and  $A_1 = \{1\}$ ,  $A_2 = \{2\}$ ,  $A_3 = \{6\}$ ,  $A_4 = \{3\}$ ,  $A_5 = \{4\}$  and  $A_6 = \{5\}$ . So the  $A_i$ 's are disjoints, and

$$X(\omega) = \sum_{i=1}^{6} x_i \mathbf{1}_{A_i}(\omega).$$

The k in the beginning of the proof correspond here to 3. So the list  $x_1, \ldots, x_k$  of distinct elements in the proof is  $x_1, x_2, x_3$  here and correspond to 2, 3, 5. As you can see, adding any  $x_i$  for i > 3 in  $x_1, x_2, x_3$  will add an element that is already in  $x_1, x_2, x_3$ . For example, if I want to add  $x_5 = 2$ , then 2 will appear twice (since  $x_1 = 2$ ). So, the list  $x_1, x_2, x_3$  is maximal in the sense that adding any element of  $x_4, x_5, x_6$  will add an element that already exist in  $x_1, x_2, x_3$ . Now,

$$J_1 = \{i \in \{1, \dots, 6\} \mid x_i = x_1\} = \{i \in \{1, \dots, 6\} \mid x_i = 2\} = \{1, 3, 4\},\$$

since  $x_1 = x_4 = x_5 = 2$ . Same,

$$J_2 = \{i \in \{1, \dots, 6\} \mid x_i = x_2\} = \{i \in \{1, \dots, 6\} \mid x_i = 3\} = \{2, 5\},\$$

since  $x_2 = x_5 = 3$ . Finally,

$$J_3 = \{i \in \{1, \dots, 6\} \mid x_i = x_3\} = \{i \in \{2, \dots, 6\} \mid x_i = 5\} = \{3\}.$$

As you can remark, the  $J_i$ 's are disjoints and

$$J_1 \cup J_2 \cup J_3 = \{1, \dots, 6\}. \tag{2}$$

Now we set,

$$A_{J_1} = A_1 \cup A_3 \cup A_4$$
,  $A_{J_2} = A_2 \cup A_5$ , and  $A_{J_3} = A_3$ .

We can then write X as the sum

$$\begin{split} X(\omega) &= 2 \cdot (\mathbf{1}_{\{1\}}(\omega) + \mathbf{1}_{\{3\}}(\omega) + \mathbf{1}_{\{4\}}(\omega)) + 3 \cdot (\mathbf{1}_{\{2\}}(\omega) + \mathbf{1}_{\{5\}}(\omega)) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{\{1,3,4\}}(\omega) + 3 \cdot \mathbf{1}_{\{2,5\}}(\omega) + 5 \cdot \mathbf{1}_{\{6\}}(\omega) \\ &= 2 \cdot \mathbf{1}_{A_{J_1}} + 3 \cdot \mathbf{1}_{A_{J_2}} + 5 \cdot \mathbf{1}_{A_{J_5}}, \end{split}$$

and since

$$A_{J_1} = \{X = 2\}, \quad A_{J_2} = \{X = 3\} \quad \text{and} \quad A_{J_3} = \{X = 5\},$$
 (3)

we finally get

$$X(\omega) = 0 \cdot \mathbf{1}_{\{X=1\}}(\omega) + 2 \cdot \mathbf{1}_{\{X=2\}}(\omega) + 3 \cdot \mathbf{1}_{\{X=3\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=4\}}(\omega) + 5 \cdot \mathbf{1}_{\{X=5\}}(\omega) + 0 \cdot \mathbf{1}_{\{X=6\}}(\omega).$$

By definition of the expectation, we have

$$\mathbb{E}[X] = 0 \cdot \mathbb{P}\{X = 1\} + 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 0 \cdot \mathbb{P}\{X = 4\} + 5 \cdot \mathbb{P}\{X = 5\} + 0 \cdot \mathbb{P}\{X = 6\}$$

$$= 2 \cdot \mathbb{P}\{X = 2\} + 3 \cdot \mathbb{P}\{X = 3\} + 5 \cdot \mathbb{P}\{X = 5\}$$

$$= 2 \cdot \mathbb{P}(A_{J_1}) + 3 \cdot \mathbb{P}(A_{J_2}) + 5 \cdot \mathbb{P}(A_{J_3}).$$

Now, since the  $A_i$ 's are disjoints, we have that

$$\mathbb{P}(A_{J_1}) = \mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4)$$
 and  $\mathbb{P}(A_{J_2}) = \mathbb{P}(A_2) + \mathbb{P}(A_5)$ ,

so

$$\begin{split} \mathbb{E}[X] &= 2 \cdot \left( \mathbb{P}(A_1) + \mathbb{P}(A_3) + \mathbb{P}(A_4) \right) + 3 \cdot \left( \mathbb{P}(A_2) + \mathbb{P}(A_5) \right) + 5 \cdot \mathbb{P}(A_6) \\ &= \left( 2 \cdot \mathbb{P}(A_1) + 2 \cdot \mathbb{P}(A_4) + 2 \cdot \mathbb{P}(A_5) \right) + \left( 3 \cdot \mathbb{P}(A_2) + 3 \cdot \mathbb{P}(A_5) \right) + 5 \cdot \mathbb{P}(A_6) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} 2\mathbb{P}(A_j) \\ &= \sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) \end{split}$$

But remember that if  $j \in J_i$ , then  $x_j = x_i$ . For example, for  $j \in J_1$ , we have

$$\sum_{j \in J_1} x_1 \mathbb{P}(A_j) = \sum_{j \in \{1,3,4\}} x_1 \mathbb{P}(A_j) = x_1 \mathbb{P}(A_1) + x_1 \mathbb{P}(A_3) + x_1 \mathbb{P}(A_4). \tag{4}$$

But since  $x_1 = x_3 = x_4 = 2$ , the right hand side of (4) is nothing more than

$$x_1 \mathbb{P}(A_1) + x_3 \mathbb{P}(A_3) + x_4 \mathbb{P}(A_4) = \sum_{j \in \{1,3,4\}} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j).$$

So finally,

$$\sum_{i=1}^{3} \sum_{j \in J_i} x_i \mathbb{P}(A_j) = \sum_{i=1}^{3} \sum_{j \in J_i} x_j \mathbb{P}(A_j) = \sum_{j \in J_1} x_j \mathbb{P}(A_j) + \sum_{j \in J_2} x_j \mathbb{P}(A_j) + \sum_{j \in J_3} x_j \mathbb{P}(A_j)$$
 (5)

and since the  $J_i$ 's are disjoints, the right hand side of (5) is

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j).$$

But as (2),  $J_1 \cup J_2 \cup J_3 = \{1, \dots, 6\}$  and thus

$$\sum_{j \in J_1 \cup J_2 \cup J_3} x_j \mathbb{P}(A_j) = \sum_{j=1}^6 x_j \mathbb{P}(A_j).$$

Finally, we conclude that  $\mathbb{E}[X]$  is

$$\mathbb{E}[X] = \sum_{i=1}^{6} x_i \mathbb{P}(A_i),$$

as wished.

### Problem 7

The prove goes by induction. For n = 1, the result is obviously correct. I recall that a function  $g : \mathbb{R} \longrightarrow \mathbb{R}$  is convex if for all  $x, y \in \mathbb{R}$ , and all  $\lambda \in [0, 1]$ ,

$$g(\lambda x + (1 - \lambda)y) \le \lambda g(x) + (1 - \lambda)g(y).$$

Suppose that

$$g\left(\sum_{i=1}^{n} a_i x_i\right) \le \sum_{i=1}^{n} a_i g(x_i),$$

for  $\sum_{i=1}^{n} a_i = 1$ . Let  $\lambda_1, \dots, \lambda_{n+1} \ge 0$  s.t.  $\sum_{i=1}^{n+1} \lambda_i = 1$ . Remark that

$$\sum_{i=1}^{n+1} \lambda_i x_i = (1 - \lambda_{n+1}) \underbrace{\frac{\lambda_1 x_1 + \dots + \lambda_n x_n}{1 - \lambda_{n+1}}}_{=:y} + \lambda_{n+1} x_{n+1} = (1 - \lambda_{n+1}) y + \lambda_{n+1} x_{n+1}.$$

Then

$$g\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = g\left((1 - \lambda_{n+1})y + \lambda_{n+1} x_{n+1}\right) \le (1 - \lambda_{n+1})g(y) + \lambda_{n+1}g(x_{n+1}),\tag{6}$$

where (1) come from the definition of the convexity. If we set  $a_i = \frac{\lambda_i}{1 - \lambda_{n+1}}$ , then

$$y = a_1 x_1 + \ldots + a_n x_n$$

and

$$\sum_{i=1}^{n} a_i = \frac{\lambda_1 + \ldots + \lambda_n}{1 - \lambda_{n+1}} = \frac{1 - \lambda_{n+1}}{1 - \lambda_{n+1}} = 1.$$

Therefore, by recurrence hypothesis,

$$g(y) = \sum_{i=1}^{n} g(x_i)a_i,$$

and thus

$$(1 - \lambda_{n+1}) \sum_{i=1}^{n} a_i g(x_i) = \sum_{i=1}^{n} \lambda_i g(x_i).$$
 (7)

Combine (6) and (7) gives

$$g\left(\sum_{i=1}^{n+1} a_i x_i\right) \le \sum_{i=1}^{n+1} a_i g(x_i).$$