Probability : Sheet 1 (solution)

Problem 1

1. The experience can be describe by the sample space $\Omega = \big\{(i,j) \mid i \in \{H,T\}, j \in \{1,\dots,6\}\big\}$ (where H is for « Head » and T for « Tail ». And the probability measure is

$$\mathbb{P}\{(i,j)\} = \frac{1}{|\Omega|} = \frac{1}{2\cdot 6}.$$

We have that

$$A = \big\{(H,i) \mid i \in \{1,\dots,6\}\big\},$$

and

$$B = \big\{(i,j) \mid i \in \{H,T\}, j \in \{1,3,5\}\big\}.$$

We have that

$$A \cap B = \big\{(H,i) \mid i \in \{1,3,5\}\big\},$$

and thus,

$$\mathbb{P}(A\cap B) = \frac{|A\cap B|}{|\Omega|} = \frac{3}{12} = \frac{1}{4}.$$

2. The sample space is

$$\Omega = \big\{\{i,j\} \mid i,j \in \{D_1, D_2, Q_1, \dots, Q_8\}\big\},$$

where Q_i is a high quality product and D_i a defective product. The probability measure is given by

$$\mathbb{P}\{i,j\} = \frac{1}{|\Omega|} = \frac{1}{\binom{10}{2}}.$$

The event « *One high quality and one defective item* »is given by

$$A := \big\{\{i,j\} \mid i \in \{D_1,D_2\}, j \in \{Q_1,\dots,Q_8\}\big\},$$

and thus

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{2}{1}\binom{8}{1}}{\binom{10}{2}}.$$

Problem 2

1. Consider the events C : « *Having a car* » and B : « *Having a bicycle* ». The sample space is

$$\Omega = \{C\cap B^c, B\cap C^c, C\cap B, C^c\cap B^c\},$$

and the probability measure is given by

$$\mathbb{P}(B) = 0.5, \quad \mathbb{P}(C) = 0.2 \quad \text{and} \quad \mathbb{P}(B\cap C) = 0.1.$$

By a formulae from the lecture,

$$\mathbb{P}(B\cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B\cap C) = 0.6.$$

2. Consider the event C :« *Infected with Chicken pos* » and M : « *Contract measles* ». The sample space is given by

$$\Omega = \{C\cap M, C^c\cap M, M^c\cap C, M^c\cap C^c\}.$$

The probability measure is

$$\mathbb{P}(C) = 0.75, \quad \mathbb{P}(M) = 0.24 \quad \text{and} \quad \mathbb{P}(M\cap C) = 0.18.$$

By a formulae from the lecture,

$$\mathbb{P}(C\cup M) = \mathbb{P}(M) + \mathbb{P}(C) - \mathbb{P}(M\cap C) = 0.81.$$

Problem 3

I recall how to compute the cardinality of the set

$$\mathcal{K}_m = \{\ell \in [1, 1000] : m \mid \ell\}$$

where $m \mid \ell$ means *"m divide ℓ"*.

$$\begin{aligned}\mathcal{K}_m &= \{km \mid k \in \mathbb{Z}, 1 \leq km \leq 1000\} \\ &= \left\{km \mid k \in \mathbb{Z}, 1 \leq k \leq \frac{1000}{m}\right\} \\ &= m\left\{k \in \mathbb{Z} \mid 1 \leq k \leq \frac{1000}{m}\right\}\end{aligned}$$

where, if A is a set, I denote

$$mA = \{ma \mid a \in A\}.$$

And thus

$$|\mathcal{K}| = \left\lfloor \frac{1000}{m} \right\rfloor, \tag{1}$$

where $\lfloor x \rfloor$ denote the biggest integer k s.t. $k \leq x$. For example, $\lfloor 2.567 \rfloor = 2$ or $\lfloor \pi \rfloor = 3$.

Example

$$|\mathcal{K}_2| = \left\lfloor \frac{1000}{2} \right\rfloor = \lfloor 500 \rfloor = 500$$

or

$$|\mathcal{K}_3| = \left\lfloor \frac{1000}{3} \right\rfloor = \lfloor 333.\bar{3} \rfloor = 333,$$

where $333, \bar{3}$ means $333, 3333333....$

Solution of problem 3

Using inclusion exclusion formulae, we directly have

$$\begin{aligned}\mathbb{P}(\mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_5) &= \mathbb{P}(\mathcal{K}_2) + \mathbb{P}(\mathcal{K}_3) + \mathbb{P}(\mathcal{K}_5) - \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_3) - \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_5) - \mathbb{P}(\mathcal{K}_3 \cap \mathcal{K}_5) + \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{K}_5) \\ &= \mathbb{P}(\mathcal{K}_2) + \mathbb{P}(\mathcal{K}_3) + \mathbb{P}(\mathcal{K}_5) - \mathbb{P}(\mathcal{K}_6) - \mathbb{P}(\mathcal{K}_{10}) - \mathbb{P}(K_{15}) + \mathbb{P}(\mathcal{K}_{30}),\end{aligned}$$

where $\mathbb{P}(\mathcal{K}_i) = \frac{|\mathcal{K}_i|}{1000}$. I let you conclude using (1).

Problem 4

Let $\{A_n\}_{n\in\mathbb{N}}$ a sequence of decreasing event, i.e. $A_{n+1} \subset A_n$ for all n. Set $B_i = A_i \setminus A_{i+1}$. Then $\{B_i\}_{i\in\mathbb{N}}$ are disjoints. Also, set $E = \bigcap_{n=1}^{\infty} A_i$. Then (make a draw if you want to be convinced),

$$A_1 = E \cup \bigcup_{i=1}^{\infty} B_i$$

and the union are disjoint. Therefore, by definition of a probability,

$$
\begin{aligned}
\mathbb{P}(A_1) &= \mathbb{P}\left(E \cup \bigcup_{i=1}^{\infty} B_i\right) \\
&\underset{dijoints\ union}{=} \mathbb{P}(E) + \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) \\
&\underset{disjoint\ union}{=} \mathbb{P}(E) + \sum_{i=1}^{\infty} \mathbb{P}(B_i) \\
&= \mathbb{P}(E) + \lim_{n\to\infty} \sum_{i=1}^{n-1} \mathbb{P}(B_i)
\end{aligned} \tag{E}
$$

Since $A_{i+1} \subset A_i$, we have that,

$$\mathbb{P}(B_i) = \mathbb{P}(A_i \setminus A_{i+1}) = \mathbb{P}(A_i) - \mathbb{P}(A_{i+1}),$$

and thus

$$\sum_{i=1}^{n-1} \mathbb{P}(B_i) = \sum_{i=1}^{n-1} \big(\mathbb{P}(A_i) - \mathbb{P}(A_{i+1})\big) = \mathbb{P}(A_1) - \mathbb{P}(A_n).$$

Therefore, the equation (E) becomes

$$\mathbb{P}(A_1) = \mathbb{P}(E) + (\mathbb{P}(A_1) - \lim_{n\to\infty} \mathbb{P}(A_n))$$

which is equivalent to

$$\lim_{n\to\infty} \mathbb{P}(A_n) = \mathbb{P}(E) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_i\right).$$

The claim follow.

Problem 6

Let consider the events K : « *He know the answer* » and C : « *The answer is correct* ». The sample space is

$$\Omega = \{K \cap C, K \cap C^c, K^c \cap C, K^c \cap C^c\},$$

and the probability measure is given by

$$\mathbb{P}(K) = p, \quad \mathbb{P}(C \mid K) = 1 \quad \text{and} \quad \mathbb{P}(C \mid K^c) = \frac{1}{N}.$$

By definition,

$$\mathbb{P}(K \mid C) = \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C \mid K)\mathbb{P}(K)}{\mathbb{P}(C)}.$$

Using Bayes formula,

$$\mathbb{P}(C) = \mathbb{P}(C \mid K)\mathbb{P}(K) + \mathbb{P}(C \mid K^c)\mathbb{P}(K^c) = p + \frac{(1-p)}{N}.$$

Finally,

$$\mathbb{P}(K \mid C) = \frac{p}{p + \frac{1-p}{N}}.$$

Problem 7

Let R_i :« *We take a red ball at the* i^{th} *thrown* ». The sample space is

$$\Omega = \{R_1 \cap R_2, R_1^c \cap R_2, R_2^c \cap R_1, R_1^c \cap R_2^c\},$$

with the probability measure

$$\mathbb{P}(R_1) = \frac{r}{r+w} \quad \mathbb{P}(R_2 \mid R_1) = \frac{r+d}{r+d+w} \quad \text{and} \quad \mathbb{P}(R_2 \mid R_1^c) = \frac{r}{r+d+w}.$$

By Bayes formulae,

$$\mathbb{P}(R_2) = \mathbb{P}(R_2 \mid R_1)\mathbb{P}(R_1) + \mathbb{P}(R_2 \mid R_1^c)\mathbb{P}(R_1^c) = \frac{r+d}{r+d+w} \cdot \frac{r}{r+d} + \frac{r}{r+d+w} \cdot \frac{w}{r+d}.$$

By definition of conditional expectation :

$$\mathbb{P}(R_1 \mid R_2) = \frac{\mathbb{P}(R_1 \cap R_2)}{\mathbb{P}(R_2)} = \frac{\mathbb{P}(R_2 \mid R_1)\mathbb{P}(R_1)}{\mathbb{P}(R_2)} = ...$$

Problem 8

Consider the events G :« *The suspect is guilty* »and C :« *The person has the caracteristic* ». By assumption, we have

$$\mathbb{P}(G) = 0.4, \quad \mathbb{P}(C) = 0.2, \quad \text{and} \quad \mathbb{P}(C \mid G) = 0.4.$$

What we want to compute is $\mathbb{P}(G \mid C)$. By definition

$$\mathbb{P}(G \mid C) = \frac{\mathbb{P}(G \cap C)}{\mathbb{P}(C)}.$$

By definition,

$$\mathbb{P}(C \mid G) = \frac{\mathbb{P}(G \cap C)}{\mathbb{P}(G)},$$

and thus

$$\mathbb{P}(G \cap C) = \mathbb{P}(C \mid G)\mathbb{P}(G) = 0.4 \times 0.4 = 0.16.$$

Therefore,

$$\mathbb{P}(G \mid C) = \frac{0.16}{0.2} = 0.8.$$