

Probability : Sheet 1 (solution)

Problem 1

1. The experience can be describe by the sample space $\Omega = \{(i, j) \mid i \in \{H, T\}, j \in \{1, \dots, 6\}\}$ (where H is for « Head » and T for « Tail »). And the probability measure is

$$\mathbb{P}\{(i, j)\} = \frac{1}{|\Omega|} = \frac{1}{2 \cdot 6}.$$

We have that

$$A = \{(H, i) \mid i \in \{1, \dots, 6\}\},$$

and

$$B = \{(i, j) \mid i \in \{H, T\}, j \in \{1, 3, 5\}\}.$$

We have that

$$A \cap B = \{(H, i) \mid i \in \{1, 3, 5\}\},$$

and thus,

$$\mathbb{P}(A \cap B) = \frac{|A \cap B|}{|\Omega|} = \frac{3}{12} = \frac{1}{4}.$$

2. The sample space is

$$\Omega = \{(i, j) \mid i, j \in \{D_1, D_2, Q_1, \dots, Q_8\}\},$$

where Q_i is a high quality product and D_i a defective product. The probability measure is given by

$$\mathbb{P}\{(i, j)\} = \frac{1}{|\Omega|} = \frac{1}{\binom{10}{2}}.$$

The event « One high quality and one defective item » is given by

$$A := \{(i, j) \mid i \in \{D_1, D_2\}, j \in \{Q_1, \dots, Q_8\}\},$$

and thus

$$\mathbb{P}(A) = \frac{|A|}{|\Omega|} = \frac{\binom{2}{1} \binom{8}{1}}{\binom{10}{2}}.$$

Problem 2

1. Consider the events C : « Having a car » and B : « Having a bicycle ». The sample space is

$$\Omega = \{C \cap B^c, B \cap C^c, C \cap B, C^c \cap B^c\},$$

and the probability measure is given by

$$\mathbb{P}(B) = 0.5, \quad \mathbb{P}(C) = 0.2 \quad \text{and} \quad \mathbb{P}(B \cap C) = 0.1.$$

By a formulae from the lecture,

$$\mathbb{P}(B \cup C) = \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(B \cap C) = 0.6.$$

2. Consider the event C : « *Infected with Chicken pos* » and M : « *Contract measles* ». The sample space is given by

$$\Omega = \{C \cap M, C^c \cap M, M^c \cap C, M^c \cap C^c\}.$$

The probability measure is

$$\mathbb{P}(C) = 0.75, \quad \mathbb{P}(M) = 0.24 \quad \text{and} \quad \mathbb{P}(M \cap C) = 0.18.$$

By a formulae from the lecture,

$$\mathbb{P}(C \cup M) = \mathbb{P}(M) + \mathbb{P}(C) - \mathbb{P}(M \cap C) = 0.81.$$

Problem 3

I recall how to compute the cardinality of the set

$$\mathcal{K}_m = \{\ell \in [1, 1000] : m \mid \ell\}$$

where $m \mid \ell$ means " m divide ℓ ".

$$\begin{aligned} \mathcal{K}_m &= \{km \mid k \in \mathbb{Z}, 1 \leq km \leq 1000\} \\ &= \left\{ km \mid k \in \mathbb{Z}, 1 \leq k \leq \frac{1000}{m} \right\} \\ &= m \left\{ k \in \mathbb{Z} \mid 1 \leq k \leq \frac{1000}{m} \right\} \end{aligned}$$

where, if A is a set, I denote

$$mA = \{ma \mid a \in A\}.$$

And thus

$$|\mathcal{K}| = \left\lfloor \frac{1000}{m} \right\rfloor, \tag{1}$$

where $\lfloor x \rfloor$ denote the biggest integer k s.t. $k \leq x$. For example, $\lfloor 2.567 \rfloor = 2$ or $\lfloor \pi \rfloor = 3$.

Example

$$|\mathcal{K}_2| = \left\lfloor \frac{1000}{2} \right\rfloor = \lfloor 500 \rfloor = 500$$

or

$$|\mathcal{K}_3| = \left\lfloor \frac{1000}{3} \right\rfloor = \lfloor 333.\bar{3} \rfloor = 333,$$

where $333.\bar{3}$ means $333,3333333\dots$

Solution of problem 3

Using inclusion exclusion formulae, we directly have

$$\begin{aligned} \mathbb{P}(\mathcal{K}_2 \cup \mathcal{K}_3 \cup \mathcal{K}_5) &= \mathbb{P}(\mathcal{K}_2) + \mathbb{P}(\mathcal{K}_3) + \mathbb{P}(\mathcal{K}_5) - \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_3) - \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_5) - \mathbb{P}(\mathcal{K}_3 \cap \mathcal{K}_5) + \mathbb{P}(\mathcal{K}_2 \cap \mathcal{K}_3 \cap \mathcal{K}_5) \\ &= \mathbb{P}(\mathcal{K}_2) + \mathbb{P}(\mathcal{K}_3) + \mathbb{P}(\mathcal{K}_5) - \mathbb{P}(\mathcal{K}_6) - \mathbb{P}(\mathcal{K}_{10}) - \mathbb{P}(\mathcal{K}_{15}) + \mathbb{P}(\mathcal{K}_{30}), \end{aligned}$$

where $\mathbb{P}(\mathcal{K}_i) = \frac{|\mathcal{K}_i|}{1000}$. I let you conclude using (1).

Problem 4

Let $\{A_n\}_{n \in \mathbb{N}}$ a sequence of decreasing event, i.e. $A_{n+1} \subset A_n$ for all n . Set $B_i = A_i \setminus A_{i+1}$. Then $\{B_i\}_{i \in \mathbb{N}}$ are disjoint. Also, set $E = \bigcap_{n=1}^{\infty} A_n$. Then (make a draw if you want to be convinced),

$$A_1 = E \cup \bigcup_{i=1}^{\infty} B_i$$

and the union are disjoint. Therefore, by definition of a probability,

$$\begin{aligned} \mathbb{P}(A_1) &= \mathbb{P}\left(E \cup \bigcup_{i=1}^{\infty} B_i\right) \\ &\stackrel{\text{disjoints union}}{=} \mathbb{P}(E) + \mathbb{P}\left(\bigcup_{i=1}^{\infty} B_i\right) \\ &\stackrel{\text{disjoint union}}{=} \mathbb{P}(E) + \sum_{i=1}^{\infty} \mathbb{P}(B_i) \\ &= \mathbb{P}(E) + \lim_{n \rightarrow \infty} \sum_{i=1}^{n-1} \mathbb{P}(B_i) \end{aligned} \tag{E}$$

Since $A_{i+1} \subset A_i$, we have that,

$$\mathbb{P}(B_i) = \mathbb{P}(A_i \setminus A_{i+1}) = \mathbb{P}(A_i) - \mathbb{P}(A_{i+1}),$$

and thus

$$\sum_{i=1}^{n-1} \mathbb{P}(B_i) = \sum_{i=1}^{n-1} (\mathbb{P}(A_i) - \mathbb{P}(A_{i+1})) = \mathbb{P}(A_1) - \mathbb{P}(A_n).$$

Therefore, the equation (E) becomes

$$\mathbb{P}(A_1) = \mathbb{P}(E) + (\mathbb{P}(A_1) - \lim_{n \rightarrow \infty} \mathbb{P}(A_n))$$

which is equivalent to

$$\lim_{n \rightarrow \infty} \mathbb{P}(A_n) = \mathbb{P}(E) = \mathbb{P}\left(\bigcap_{n=1}^{\infty} A_n\right).$$

The claim follow.

Problem 6

Let consider the events K : « He know the answer » and C : « The answer is correct ». The sample space is

$$\Omega = \{K \cap C, K \cap C^c, K^c \cap C, K^c \cap C^c\},$$

and the probability measure is given by

$$\mathbb{P}(K) = p, \quad \mathbb{P}(C | K) = 1 \quad \text{and} \quad \mathbb{P}(C | K^c) = \frac{1}{N}.$$

By definition,

$$\mathbb{P}(K | C) = \frac{\mathbb{P}(K \cap C)}{\mathbb{P}(C)} = \frac{\mathbb{P}(C | K)\mathbb{P}(K)}{\mathbb{P}(C)}.$$

Using Bayes formula,

$$\mathbb{P}(C) = \mathbb{P}(C | K)\mathbb{P}(K) + \mathbb{P}(C | K^c)\mathbb{P}(K^c) = p + \frac{(1-p)}{N}.$$

Finally,

$$\mathbb{P}(K | C) = \frac{p}{p + \frac{1-p}{N}}.$$

Problem 7

Let R_i : « We take a red ball at the i^{th} throw ». The sample space is

$$\Omega = \{R_1 \cap R_2, R_1^c \cap R_2, R_2^c \cap R_1, R_1^c \cap R_2^c\},$$

with the probability measure

$$\mathbb{P}(R_1) = \frac{r}{r+w} \quad \mathbb{P}(R_2 | R_1) = \frac{r+d}{r+d+w} \quad \text{and} \quad \mathbb{P}(R_2 | R_1^c) = \frac{r}{r+d+w}.$$

By Bayes formulae,

$$\mathbb{P}(R_2) = \mathbb{P}(R_2 | R_1)\mathbb{P}(R_1) + \mathbb{P}(R_2 | R_1^c)\mathbb{P}(R_1^c) = \frac{r+d}{r+d+w} \cdot \frac{r}{r+d} + \frac{r}{r+d+w} \cdot \frac{w}{r+d}.$$

By definition of conditional expectation :

$$\mathbb{P}(R_1 | R_2) = \frac{\mathbb{P}(R_1 \cap R_2)}{\mathbb{P}(R_2)} = \frac{\mathbb{P}(R_2 | R_1)\mathbb{P}(R_1)}{\mathbb{P}(R_2)} = \dots$$

Problem 8

Consider the events G : « The suspect is guilty » and C : « The person has the characteristic ». By assumption, we have

$$\mathbb{P}(G) = 0.4, \quad \mathbb{P}(C) = 0.2, \quad \text{and} \quad \mathbb{P}(C | G) = 0.4.$$

What we want to compute is $\mathbb{P}(G | C)$. By definition

$$\mathbb{P}(G | C) = \frac{\mathbb{P}(G \cap C)}{\mathbb{P}(C)}.$$

By definition,

$$\mathbb{P}(C | G) = \frac{\mathbb{P}(G \cap C)}{\mathbb{P}(G)},$$

and thus

$$\mathbb{P}(G \cap C) = \mathbb{P}(C | G)\mathbb{P}(G) = 0.4 \times 0.4 = 0.16.$$

Therefore,

$$\mathbb{P}(G | C) = \frac{0.16}{0.2} = 0.8.$$