Random Matrices

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1 What is a random matrix and possible applications

1.1 Setup of notation : Random variable (r.v.)

- For an interval $\mathfrak{S} \subset \mathbb{R}$ we denote $\rho(x)$ it's PDF (Probability Density Function).

$$\int_a^b \rho(x)\,\mathrm{d}x = \mathbb{P}\{X \in [a,b] \subset \mathfrak{S}\}.$$

- If $\mathbb{P}\{X \in \mathfrak{S}\} = 1$, the average of X (or 1^{st} moment)

$$\langle X \rangle := \int_{\mathfrak{S}} x\rho(x)\,\mathrm{d}x,$$

and the n^{th} moment is given by

$$\langle X^n \rangle := \int_{\mathfrak{S}} \mathrm{x}^n \rho(x)\,\mathrm{d}x.$$

- The variance is given by

$$\mathrm{Var}(X) = \left\langle (X - \langle X \rangle)^2 \right\rangle = \left\langle X^2 \right\rangle - \langle X \rangle^2 .$$

A variable is centered if $\langle X \rangle = 0$, and thus $\mathrm{Var}(X) = \left\langle X^2 \right\rangle$. For example Gauss r.v.'s with PDF

$$\rho(x) = \sqrt{\frac{a}{\pi}} e^{-ax^2},$$

$\mathfrak{S} = \mathbb{R}$, $\langle X \rangle = 0$ and $\mathrm{Var}(X) = \frac{1}{2a}$.

- The Cumulative Distribution Function (CDF) is defined as

$$F(x) = \int_{-\infty}^{x} \rho(x)\,\mathrm{d}x.$$

We have

$$\lim_{x\to-\infty} F(x) = 0 \quad \text{and} \quad \lim_{x\to+\infty} F(x) = 1.$$

- For $n \geq 2$, random variables $X_1, \ldots, X_n$ are described by the Joint PDF (JPDF) $\rho(x_1, \ldots, x_n)$. So $\rho(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n$ is the probability to find

$$X_1 \in [x_1, x_1 + \mathrm{d}x_1], \ldots, X_n \in [x_n, x_n + \mathrm{d}x_n].$$

- The r.v.'s are independent if

$$\rho(x_1, \ldots, x_n) = \prod_{i=1}^{n} \rho(x_i).$$

- The marginal PDF of X_1 is given by

$$\rho(x_1) := \int \rho(x_1, \ldots, x_n)\,\mathrm{d}x_2 \ldots \mathrm{d}x_n.$$

It's the probability that

$$X_1 \in [x_1, x_1 + \mathrm{d}x_1],$$

independently of all others r.v.'s

- Change of variables if $x_i = x_i(y)$, $i = 1, \ldots, n$ and $y = (y_1, \ldots, y_n)$, then

$$\rho(x_1, \ldots, x_n)\,\mathrm{d}x_1 \ldots \mathrm{d}x_n = \rho(x_1(y), \ldots, x_n(y))|\mathcal{J}(x,y)|\,\mathrm{d}y_1 \ldots \mathrm{d}y_n,$$

with $\mathcal{J}(x,y) = \det\left(\frac{\partial x_j}{\partial y_i}\right)_{1\leq i,j\leq n}$ the Jacobian matrix.

1.2 What is a random matrix ?

1.3 The Gaussian Orthogonal Ensemble GOE

Take a matrix H of size $N \times N$, and fill in r.v.'s $H_{i,j}$, $i, j = 1, \ldots, N$ that are independent and Gaussian $\mathcal{N}(0,1)$. In general, the matrix H that we have constructed is not symmetric ($H \neq H^T$). We symmetrize it :

$$H_s = \frac{1}{2}(H + H^T).$$

By linear algebra, we can write H_s as

$$H_s = O \Lambda O^T,$$

where $O \in \mathcal{O}(N)$ and $\Lambda = \mathrm{Diag}(x_1, \ldots, x_N)$.

Remark 1.

1. We'll not consider matrices with complex eigenvalues.

2. This H_s is one member of the **GOE**.

There are other ensembles : Gaussian Unitary Ensembles (**GUE**). Suppose $\tilde{H}_{ij}$ has real and imaginary part in $\mathcal{N}(0,1)$ and

$$\tilde{H}_{herm} := \frac{1}{2}(\tilde{H} + \tilde{H}^\dagger).$$

1.3.1 Distribution of matrix elements

GOE probability measure on the set of random matrix of size $N \times N$ is given by

$$\rho(H) = \prod_{1 \leq i,j \leq N} \rho(H_{ij}) = \prod_{1 \leq i,j \leq n} \frac{1}{\sqrt{2\pi}} e^{\frac{-H_{ij}^2}{2}},$$

where $H = (H_{ij}) \in \mathcal{S}(N)$ with independent Gaussian r.v.'s entries.

The eigenvalues of a non-symmetric matrix with real entries are complex-conjugated in the sense that if λ is an eigenvalue, then so is $\bar{\lambda}$. That characteristic polynomial of H is given by

$$p(\lambda) = \det(\lambda I_n - H),$$

and has real coefficient. We denote the set of square random matrices with real Gaussian's independent r.v.'s entries as the *Real Ginibre Ensemble*. In the **GOE**,

$$H_s = \frac{1}{2}(H + H^T).$$

We have N diagonal elements $(H_s)_{ii}$, $i = 1, \ldots, N$ and $\frac{N(N-1)}{2}$ upper triangular part $(H_s)_{ij}$, with $i < j$ and $i, j = 1, \ldots, N$.

Since $H_{ij} \sim \mathcal{N}(0,1)$ for all i, j and that there are independent, we have that

$$(H_s)_{ij} = \frac{1}{2}(H_{ij} + H_{ji}) \sim \mathcal{N}\left(0, \frac{1}{2}\right), \quad i \neq j.$$

Therefore

$$
\begin{aligned}
\rho(H_s) = \rho((H_s)_{1\le i\le j\le N}) &= \prod_{i=1}^{N} \frac{e^{\frac{-(H_s)_{ii}^2}{2}}}{\sqrt{2\pi}} \prod_{\substack{1\le j\le N\\ i<j}} \frac{e^{-(H_s)_{ij}^2}}{\sqrt{\pi}} \\
&= \frac{1}{2^{\frac{N}{2}}\pi^{\frac{N^2}{2}}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}(H_s)_{ii}^2 - \sum_{1\le i<j\le N}(H_s)_{ij}^2\right\} \\
&= \frac{1}{2^{\frac{N}{2}}\pi^{\frac{N^2}{2}}} \exp\left\{-\frac{1}{2}\sum_{i=1}^{N}(H_s)_{ii}^2 - \frac{1}{2}\sum_{\substack{1\le i,j\le N\\ i\ne j}}(H_s)_{ij}^2\right\} \\
&= \frac{1}{2^{\frac{N}{2}}\pi^{\frac{N^2}{2}}} \exp\left\{-\frac{1}{2}\operatorname{Tr}(H_s^2)\right\},
\end{aligned}
$$

since

$$\sum_{1\le i,j\le N}(H_s)_{ij}(H_s)_{ji} = \operatorname{Tr}(H_s H_s^T),$$

and H_s is symmetric. Similarly, for H in the Real Ginibre ensemble (i.e. H is not necessarily symmetric),

$$\rho(H) \propto \exp\left\{-Tr(HH^T)\right\}.$$

Outcome

$$Tr(H_s^2) = \sum_{i=1}^{N} X_i^2,$$

where the X_i's are the eigenvalues of H. The Goal is to understand statistics of **GOE** eigenvalues. From $\rho(H_s)$ we derive the JPDF of the eigenvalues given by $\rho(x_1,\ldots,x_n)$. We'll show latter that

$$\rho(x_1,\ldots,x_N) = \frac{1}{Z_{N,B}} e^{-\frac{1}{2}\sum_{i=1}^{N} x_i^2} \prod_{1\le j<k\le N} |x_k - x_j|^\beta,$$

and thus, that the $X_i' s$ are not independent. The term $|x_k - x_j|$ come from $H_s = O\Lambda O^T$ and

$$Z_{N,\beta} = (2\pi)^{N/2}\prod_{j=1}^{N}\frac{\Gamma(1+j\beta/2)}{\Gamma(1+\beta/2)},$$

where for **GOE** we have $\beta = 1$, for **GUE** we have $\beta = 2$ and for **GSE** (Gaussian Sympletic Ensemble) we have $\beta = 4$. Therefore,

$$\rho(x_1,\ldots,x_n) \propto \exp\left\{-\frac{1}{2}\sum_{i=1}^{N} x_i^2 + \beta\sum_{j<k}\log|x_k - x_j|\right\},$$

for $\beta > 0$ and Gaussian β ensemble. Eigenvalues behave like N charged particules interactiong with Coulomb repulsion in 2D at certains values of inverse temperature β while being confined by a Gauss potential.

A popular quantity : level spacing distribution $P(s)$ which is the probability to find two consecutive eigenvalues at distance s. For **GOE** of $N \times N$ matrices

$$P_{\mathbf{GOE}}(s) = \frac{s}{2}e^{-s^2/4}.$$

Remark 2. Two surprises arises :

1. Extreme by accurate approximate for $N \times N$ **GOE**,

2. Can be found in many phenomena in Nature.