Dynamics of interacting systems

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Contents

1 Deterministic dynamical systems in $\mathbb{R}^n$

Let consider the ODE

$$\begin{cases} \dot{x}_t = f(x_t, t; \mu) \\ x_{t_0} = x_0 \end{cases}$$

For now, μ is not revelent for us. We write $\dot{x} = f(x,t)$.

Théorème 1.1 (Local existence and uniqueness)**.**

Assume $U \subset \mathbb{R}^n$ open, and $f : U \longrightarrow \mathbb{R}^n$ locally Lipschitz, i.e. for all open set $\mathcal{O} \subset U$ s.t. $\overline{\mathcal{O}}$ is compact, there is $\gamma = \gamma(\mathcal{O})$ s.t.

$$\|f(x) - f(y)\| \leq \gamma \|x - y\|,$$

for all $x, y \in \mathcal{O}$. Then, for all $x_0 \in U$, there is a $r > 0$ and a unique $(x_t)_{t \in (-r,r)}$ s.t. $(x_t)_t$ satisfy the autonomous ODE $\dot{x} = f(x)$ with initial condition x_0.

Example

The theorem hold locally only. For example $\dot{x} = 1 + x^2$ has solution $x_t = \tan(t + c)$ (where c is determinate by initial conditions). The solution leaves any bounded set in finite time.

Given initial condition (t_0, x_0), we write (as needed) :

$$x_t = x(t) = x(t, t_0, x_0) = x(t, t_0, x_0; \mu) = x(t; \mu).$$

We describe here different notions :

1. The *phase space*. Here $\mathbb{R}^n$, but it can also be a cylinder, a sphere, any compact manifold...
2. The *vector field* $f(x,t,\mu)$ describes tangent vector at the point x_t of the solution curve $x = (x_t)_t$,
3. $\mathcal{O}(x_0) := \{x(t, t_0, x_0) \mid t \in I\}$ is the *orbit* through $x_0 \in U$, i.e. the points in phase space which lie on the solution curve / trajectory passing through x_0. Note that $\mathcal{O}(x_0) = \mathcal{O}\big(x(T, t_0, x_0)\big)$ for all $T \in I$.

Example

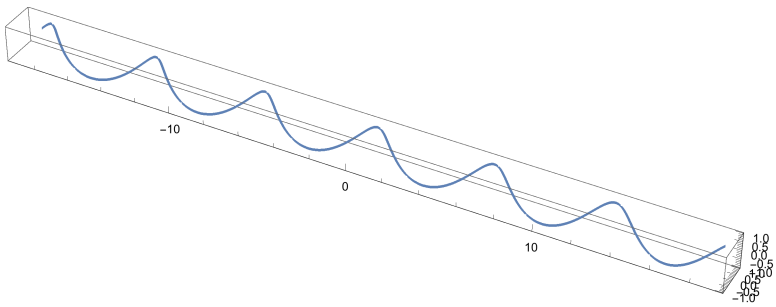
Consider

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad (x,y) \in \mathbb{R}^2,$$

and the initial condition $(x_0, y_0) = (1, 0)$. Then, $(x_t, y_t) = (\cos(t), -\sin(t))$. The graph of trajectory (or integral curves) passing through $(1,0)$ at $t = 0$ are

$$\{(\cos(t), -\sin(t), t) \mid t \in \mathbb{R}\}.$$

We see on the next picture



The Orbit through $(1,0)$ is

$$\{(\cos(t), -\sin(t)) \mid t \in \mathbb{R}\} = \{(x,y) \in \mathbb{R}^2 \mid x^2 + y^2 = 1\}.$$

As opposed to the graph, the orbit is merely a set of points in phase space.

Let define $\Phi^{t,t_0}(x_0) := x(t, t_0, x_0)$ or for autonomous systems $\dot{x} = f(x)$, $\Phi^t(x_0) := x(t, t_0)$. Let consider autonomous vector field $\dot{x} = f(x)$.

Définition 1.2 (stationary point).

The point $\bar{x} \in U$ is called a stationary point *if* $f(\bar{x}) = 0$.

If a stationary point $\bar{x}$ is an initial condition, then the solution of $\dot{x} = f(x)$ doesn't change over time.

1.1 Stability of trajectories

Let $\bar{x} = (\bar{x}_t)_t$ be a solution of $\dot{x} = f(x)$.

Définition 1.3 (Lyapunov stability / Local stability).

A trajectory $\bar{x}$ is called stable *if*

$$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0 : \forall \xi, \|\bar{x}(t_0) - \xi\| < \delta \implies \forall t > t_0, \|\bar{x}(t) - \Phi^{t,t_0}(\xi)\| = \|\Phi^{t,t_0}(\bar{x}(t_0)) - \Phi^{t,t_0}(\xi)\| < \varepsilon$$

where ξ above is an initial condition.

Remark 1.

1. $\|\cdot\|$ is any norm in $\mathbb{R}^n$,
2. Existence of solution for all t is assumed implicitly,
3. Concept for trajectories : if $\hat{x}$ is a stationary point, the concept can by applied as well, using that $\hat{x} = \Phi^{t,t_0}(\hat{x})$ for all t.
4. A trajectory is stable if all solutions starting in a sufficently small neighborhood stay close for ever,
5. If $\dot{x} = f(x)$ has a unique solution given an arbitrary initial condition, then a stationary point cannot be reached in finite time.

Homework

Does the following system has Lyapunov stable trajectories ?

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

Définition 1.4 (Unstable trajectories).

Any trajectory that is not stable is called unstable

Example

A saddle

$$\begin{cases} \dot{x} = x \\ \dot{y} = -y \end{cases}, \quad (x, y) \in \mathbb{R}^2.$$

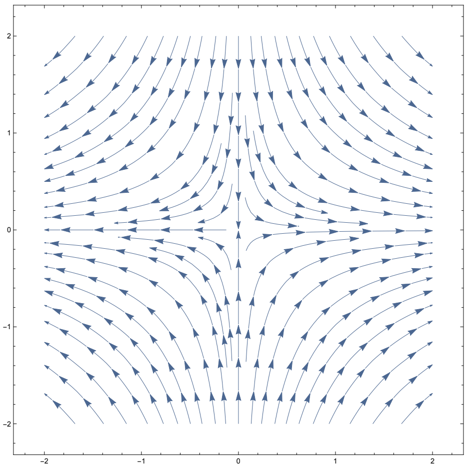
Since the system is not coupled, we may consider x and y separately :

1. $x > 0$ (resp. $x < 0$) if and only if x is locally increasing (resp. locally decreasing). Therefore, the stationary point $\bar{x} = 0$ is clearly unstable. To picture the behaviour, take the advantage of the fact that in $\mathbb{R}$, any dynamic system is a gradient system : $\dot{x} = -\nabla\left(-\frac{x^2}{2}\right)$.

2. The opposite hold true for y : $\bar{y} = 0$ is a stable stationary point; $y_t \to 0$ for all initial condition.

$\dot{y} = -\nabla\left(\frac{y^2}{2}\right)$

3. If we combined **1.** and **2.** then $(0, 0)$ is unstable.



Définition 1.5 (Asymptotic stability)**.**

The solution of $\dot{x} = f(x)$ *is asymptotically stable if*

1. $\bar{x}$ *is Lyapunov stable,*

2. *The* δ *from the definition 1.1 can be chosen small enough to guarantee*

$$\lim_{t\to\infty} \|\bar{x}(t) - \Phi^t(\xi)\| = 0$$

for all initial condition ξ *with* $\|\bar{x}(t_0) - \xi\| < \delta$.

Example

The part **2.** of the previous definition can be satisfied even if **1.** is violated. Consider the system in $\mathbb{R}^2$ in polar coordinate satisfying

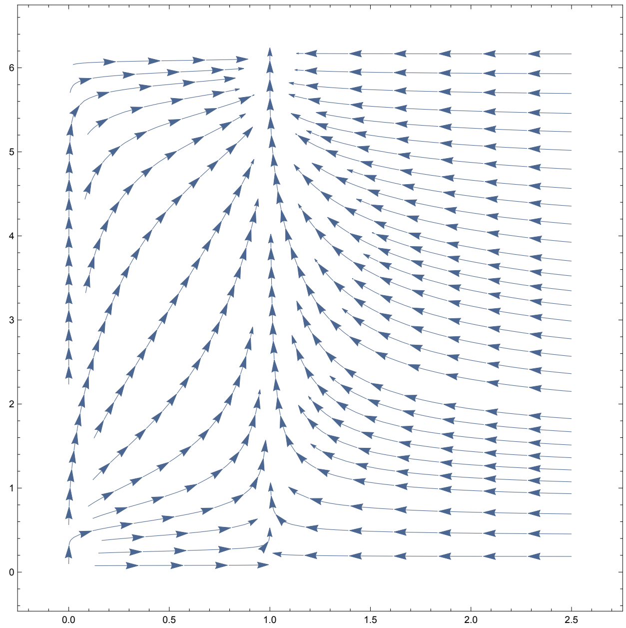
$$\begin{cases} \dot{r} = r(1 - r) & r \geq 0, \\ \dot{\theta} = \sin^2\left(\frac{\theta}{2}\right) & \theta \in [0, 2\pi). \end{cases}$$

This system is not coupled, r has a stationary point in 0 and 1. Moreover, $\dot{r} > 0 \iff r < 1$ and $\dot{r} < 0 \iff r > 1$. Thus $r_t \to 1$ unless $r_0 = 0$. Moreover, since $\theta > 0$, we have $\dot{\theta} > 0$ and thus θ is increasing. Therefore $\theta_t \to 2\pi \pmod{2\pi}$. Thus $\theta_t \equiv 0$ or $\theta_t \to 2\pi \pmod{2\pi} = 0$. We conclude that either $r_0 = 0$ and $(r_t, \theta_t) \to (0, 0)$ or $r_0 > 0$ and $(r_t, \theta_t) \to (1, 0)$. If $\theta_0 > 0$, there is a long excursion before approaching the stationary. Therefore, there is no Lyapunov stability.

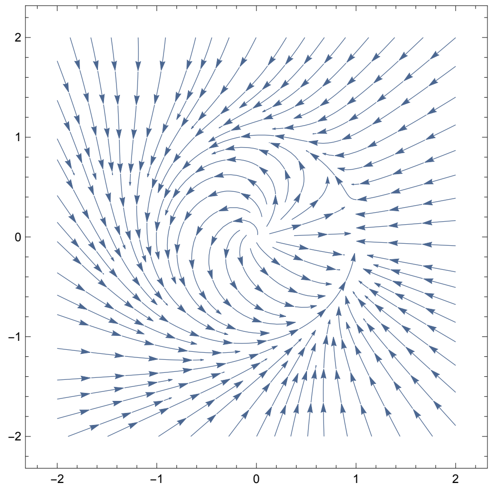
Remark 2. In cartesian coordinate, the system is given by

$$\begin{cases} \dot{x} = (1 - \sqrt{x^2 + y^2})x - \frac{1}{2}\left(1 - \frac{x}{\sqrt{x^2+y^2}}\right) y \\ \dot{y} = (1 - \sqrt{x^2 + y^2})y + \frac{1}{2}\left(1 - \frac{x}{\sqrt{x^2+y^2}}\right) x. \end{cases}$$

In pola coordinate the system is given by



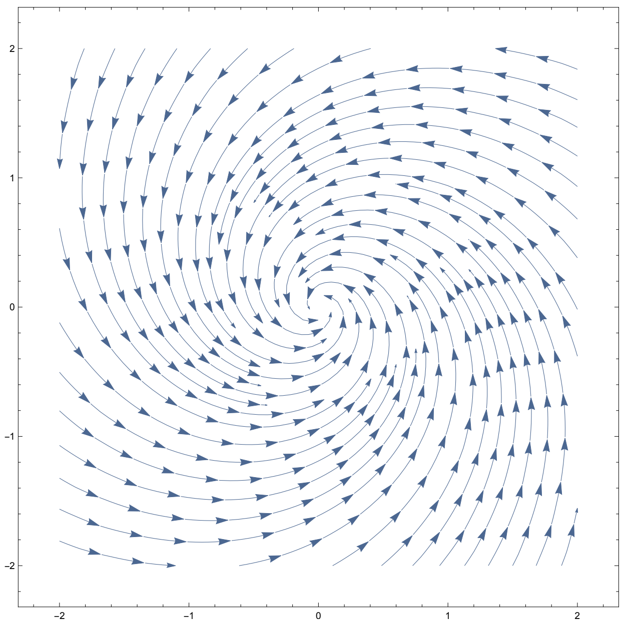
The $x-$axis represent r and the $y-$axis represent θ. In cartesian coordinate, it's given by



Example

An example of asymptotically stable system :

$$\begin{cases} \dot{x} = -x - 2y \\ \dot{y} = 2x - y \end{cases}, \quad (x, y) \in \mathbb{R}^2.$$



1.2 The use of Lyapunov functions

Basic idea : If $\bar{x}$ is a stable stationary point, then there exist a neighborhood U of $\bar{x}$ s.t. trajectories remain in U once they entered. It's true if vector field f at ∂U is pointing inwards or is tangential

This needs to remain true for smaller and smaller U shrinking to $\{\bar{x}\}$.

Théorème 1.6 (Lyapunov functions)**.**

Let $\bar{x}$ be stationary for $\dot{x} = f(x)$, W be a neighborhood of $\bar{x}$ s.t. $\bar{x} \in W \subset U$ and $\mathcal{V} : W \longrightarrow \mathbb{R}$ a differentiable function s.t. $\mathcal{V}(\bar{x}) = 0$, $\mathcal{V}(x) > 0$ for all $x \neq \bar{x}$ and $\dot{\mathcal{V}}(x) \leq 0$ on $W \setminus \{\bar{x}\}$ where

$$\dot{\mathcal{V}}(x) = \langle \nabla \mathcal{V}(x), f(x) \rangle = \frac{\mathrm{d}}{\mathrm{d}t} \mathcal{V}(x_t),$$

is the derivative of $\mathcal{V}$ along the trajectory $(x_t)_t$. Then $\bar{x}$ is stable. Moreover, if $\dot{\mathcal{V}}(x) < 0$, then $\bar{x}$ is asymptotically stable.

The function $\mathcal{V}$ is called a *Lyapunov function.*

Remark 3.

1. If the choice $W = U$ is possible, then $\bar{x}$ is globally asymptotically stable. As a consequence, all solution remain bounded and satisfy $x_t \to \bar{x}$ as $t \to \infty$. This allows to test stability and boundness without solving the ODE.
2. The problem is that there is no general method to find Lyapunov functions.
3. For multiple stationary points, find local Lyapunov functions.

Proof.

Step 1 : Let $\delta > 0$ small enough to have $\mathcal{B}_\delta(\bar{x}) \subset\subset U$[1] Set

$$m := \min\{\mathcal{V}(x) \mid \partial\mathcal{B}_\delta(\bar{x})\}.$$

Define

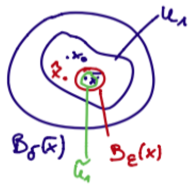
$$U_1 := \{x \in \mathcal{B}_\delta(\bar{x}) \mid \mathcal{V}(x) < m\},$$

and let $x_0 \in U_1 \setminus \{\bar{x}\}$. Since $\dot{\overline{\mathcal{V}(x)}} \leq 0$, then $\mathcal{V}(\Phi^t(x_0))$ is non decreasing. By definition of U_1, we have that $\Phi^t(x_0)$ remains in U_1 for all t, and thus in $\mathcal{B}_\delta(\bar{x})$. Since $\delta > 0$ is unspecified, this show the stability.

Step 2 : Assume now that the stronger hypothesis $\dot{\mathcal{V}}(x) < 0$ on $W \setminus \{\bar{x}\}$. Then $V(\Phi^t(x_0))$ is strictly decreasing for $x_0 \in U_1$. Let $(t_n)_n$ an increasing sequence that diverge to ∞. Since $\overline{\mathcal{B}_\delta(\bar{x})}$ is compact, $(\Phi^{t_n}(x_0))_n$ has a convergent subsequence $(\Phi^{t_{n_k}}(x_0))_k$ satisfying

$$\Phi^{t_{n_k}}(x_0) \underset{k\to\infty}{\longrightarrow} \hat{x} \in \overline{\mathcal{B}_\delta(\bar{x})}.$$

Step 3 : By contradiction, suppose $\hat{x} \neq \bar{x}$. Then, there is $\varepsilon > 0$ s.t. $\bar{x} \notin \overline{\mathcal{B}_\varepsilon(\bar{x})} \subset U_1$. As in the step 1, let $\tilde{U}_1$ a neighborhood of $\bar{x}$ s.t. $\bar{x} \in \tilde{U}_1 \subset \overline{\mathcal{B}_\varepsilon(\bar{x})}$ and $\Phi^t(\xi) \in \overline{\mathcal{B}_\varepsilon(\bar{x})}$ for all $\xi \in \tilde{U}_1$.



Consequently, $\Phi^{t_{n_k}}(x_0)$ can never enter in $\tilde{U}_1$. Thus, $\Phi^t(x_0)$ canot enter in $\tilde{U}_1$ neither, (otherwise we could choose all t_n s.t. $\Phi^{t_n}(x_0) \in \overline{\mathcal{B}_\varepsilon(\bar{x})}$ leading to a contradiction). So far, we established that $x = (x_t)_t$ start and remains forever in U_1 while avoiding $\tilde{U}_1$. Since $\dot{\mathcal{V}}(x) < 0$, we finally have that there is K s.t. $\dot{\mathcal{V}}(x) < -K$ and thus

$$\mathcal{V}(\Phi^{t_{n_k}}(x_0)) - \mathcal{V}(x_0) = \int_0^{t_{n_k}} (\dot{\mathcal{V}}(\Phi^s(x_0))\,\mathrm{d}s \leq -Kt_{n_k}.$$

Therefore

$$\mathcal{V}(\Phi^{t_{n_k}}(x_0)) \leq \mathcal{V}(x_0) - Kt_{n_k} \underset{k\to\infty}{\longrightarrow} -\infty,$$

and thus, there is k_0 s.t. for all $k \geq k_0$,

$$\mathcal{V}(\Phi^{t_{n_k}}(x_0)) < 0,$$

which is a contradiction since a Lyapunov function is non-negative.

$\square$

Example

1. Consider

$$\begin{cases} \dot{x} = y \\ \dot{y} = x + \gamma x^2 y \end{cases}, \quad (x,y) \in \mathbb{R}^2.$$

There is a stationary point at $(0,0)$. A Lyapunov function is given by

$$\mathcal{V}(x,y) = \frac{1}{2}\|(x,y)\|^2.$$

[1] $U \subset\subset V$ mean that $\overline{U} \subset V$.

It's a Lyapunov function since $V(0,0)=0$, $V(x,y)>0$ whenever $(x,y)\neq(0,0)$ and

$$\dot{\mathcal{V}}(x,y)=\left\langle \nabla\mathcal{V}(x,y),\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\right\rangle=\left\langle \begin{pmatrix}x\\ y\end{pmatrix},\begin{pmatrix}y\\ -x+\gamma x^2 y\end{pmatrix}\right\rangle=\gamma x^2 y \underset{\gamma<0}{\leq} 0.$$

Therefore, for $\gamma<0$, $(0,0)$ is globally stable. We would have to work harder to show that $(0,0)$ is even (globally) asymptotically stable (previous theorem doesn't help here).

2. A particle of mass m attached to a spring of stiffness $k(x+x^3)$. The ODE for the displacement of the particle is given by

$$m\ddot{x}+k(x+x^3)=0,\quad k>0.$$

Rewrinting with $y=\dot{x}$ gives

$$\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{k}{m}(x+x^3).\end{cases}$$

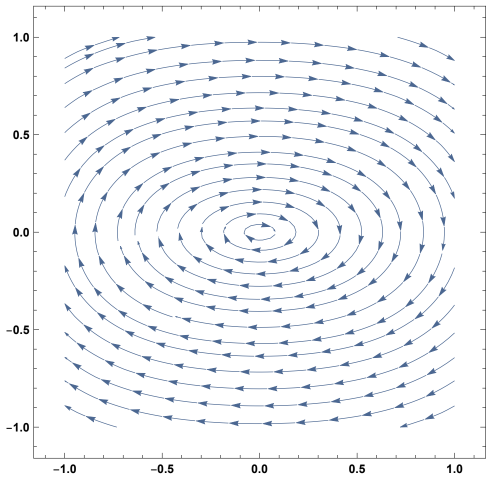
It has a unique stationary point at $(0,0)$. The total energy is given by

$$E(x,y)=m\frac{y^2}{2}+k\left(\frac{x^2}{2}+\frac{x^4}{4}\right).$$

It's a Lyapunov function since $E\geq 0$ and $E(x,y)=0$ if and aonly if $(x,y)=(0,0)$ and

$$\dot{E}(x,y)=\left\langle \nabla E(x,y),\begin{pmatrix}\dot{x}\\ \dot{y}\end{pmatrix}\right\rangle=\left\langle \begin{pmatrix}k(x+x^3)\\ my\end{pmatrix},\begin{pmatrix}y\\ -\frac{k}{m}(x+x^3)\end{pmatrix}\right\rangle=0.$$

Therefore $(0,0)$ is stable. Is $(0,0)$ asymptotically stable ? (homework). For $k=1$ et $m=4$, we have the following trajectories :



3. Same system than **2.** with a damping term added

$$\begin{cases}\dot{x}=y\\ \dot{y}=-\frac{k}{m}(x+x^3)-\alpha y.\end{cases}$$

Use the same Lyapunov function as before

$$\dot{E}(x,y)=-\alpha m y^2\quad\text{and}\quad \dot{E}(x,0)=0.$$

So we don't have information on the asymptotical stability. We modify the Lyapunov function and we consider

$$\mathcal{V}(x,y)=m\frac{y^2}{2}+k\left(\frac{x^2}{2}+\frac{x^4}{4}\right)+\beta(xy+\alpha\frac{x^2}{2}),$$

for some $\beta\in\mathbb{R}$. We have

- $\mathcal{V}(0,0) = 0$,
- $\mathcal{V}(x,y) > 0$ for $(x,y) \neq (0,0)$ and $\beta > 0$ small enough (using $2|xy| \leq x^2 + y^2$).
- $\dot{\mathcal{V}}(x,y) = -\beta\frac{k}{m}(x^2 + x^4) - (\alpha m - \beta)y^2 < 0$ for $(x,y) \neq (0,0)$ and β small.

Therefore, for $\alpha > 0$, $(0,0)$ is globally asymptotically stable. How does the vector field change when we introduce the damping $\alpha > 0$?

1.3 Further concept of stability

Définition 1.7 (Orbital stability)**.**

$\bar{x} = (\bar{x}_t)_t$ *is* orbitally stable *if*

$$\forall \varepsilon > 0, \exists \delta > 0 : \|\bar{x}(t_0) - \Phi^{t_0}(\xi)\| < \delta \implies \forall t > t_0, \operatorname{dist}(\Phi^{t,t_0}(\xi), \mathcal{O}^+(x_0, t_0)) < \varepsilon,$$

whose $\mathcal{O}^+(x_0, t_0) = \{\Phi^{t,t_0}(x_0) \mid t \geq t_0\}$ *(orbit, forward in time).*

Remark 4. In other word, if the trajectory $\Phi^t(\xi)$ is close to orbit $\mathcal{O}(x_0, t_0)$ at time t_0, then the distance of trajectory remains close to the orbit forward in time.

The idea is it allow for different speeds.

Définition 1.8 (asymptotical orbital stability)**.**

$\bar{x} = (\bar{x}_t)_t$ *is asymptotically orbitally stable if* $\bar{x}$ *is orbitally stable and*

$$\exists \delta > 0 : \forall \xi : \|\bar{x}(t_0) - \Phi^{t_0}(\xi)\| < \delta \implies \lim_{t\to\infty} \operatorname{dist}(\Phi^{t,t_0}(\xi), \mathcal{O}^+(x_0, t_0)) = 0.$$

How to test Lyapunov or asymptotic stability ? Consider the problem $\dot{x} = f(x)$. Let $f(\bar{x}) = 0$, i.e. $\bar{x}$ is a stationary solution. Let x a orbitrary solution (orbitrary ?) of $\dot{x} = f(x)$. Set $y_t := x_t - \bar{x}$, i.e.

$$\dot{y}_t = \dot{x}_t = f(x_t) = f(\bar{x} + y_t) \underset{\text{Taylor}}{=} \underbrace{f(\bar{x})}_{=0} + Df(\bar{x})y_t + \underbrace{\mathcal{R}(y_t)}_{=\mathcal{O}(\|y_t\|^2) \text{ for } f \in \mathcal{C}^2}.$$

The PDE $\dot{y}_t = \nabla f(\bar{x})y_t + \mathcal{R}(y_t)$ describes deviation from $\bar{x}$. As long as $\|y_t\|$ is small, the linearization $\dot{y}_t^0 = Df(\bar{x})y_t^0$ provides a good approximation. For linearization,

$$y_t^0 = e^{Df(\bar{x})t}y_0.$$

Two steps :

1. Is $y_t^0 = 0$ stable for the linearized equation ?

2. Sho : If $y_t^0 \equiv 0$ is stable for linear equation, then $\bar{x}$ is stable for $\dot{x} = f(x)$:

$$y^0 \text{ stable} \implies \bar{x} \text{ stable}.$$

1. Easy (Homework). If all eigenvalues λ_i of $Df(\bar{x})$ satisfy $\Re(\lambda_i) < 0$, then $y^0 = 0$ is asymptotically stable. It's obvious for $n = 1$ since $y_t^0 = y_0 e^{\lambda t} \underset{t\to\infty}{\longrightarrow} 0$ whenever $\lambda < 0$. If $\dot{x} = f(x)$ is not autonomous, even step 1 can be hood. There is no general analytical method if $Df(\bar{x}_t)$ depend on t.

2.

Théorème 1.9 (Stability in non-linear systems)**.**

Assume $\Re(\lambda_i) < 0$ *for all eigenvalue* λ_i *of* $Df(\bar{x})$*. Then* $x_t = \bar{x}$ *is asymptotically stable.*

Proof. We'll use Lyapunov function. Let $y = \varepsilon u$ with $\varepsilon \in (0, 1)$. Then

$$\begin{aligned} \dot{u} &= \frac{1}{\varepsilon}\dot{y} + \frac{1}{2}Df(\bar{x})y + \frac{1}{\varepsilon}\mathcal{R}(y) \\ &= Df(\bar{x})u + \frac{1}{\varepsilon}\mathcal{R}(\varepsilon u), \end{aligned}$$

where

$$\|\mathcal{R}(\varepsilon u)\| \leq \frac{1}{\varepsilon}\mathcal{O}(\varepsilon^2\|u\|^2) = \varepsilon\mathcal{O}(\|u\|^2).$$

Let $\mathcal{V}(u) = \frac{1}{2}\|u\|^2$. It's a Lyapunof function at $u = 0$ since $\mathcal{V}(0) = 0$, $V(u) > 0$ for $u \neq 0$ and

$$\dot{\mathcal{V}}(u) = \langle \nabla V(u), \dot{u} \rangle = \langle u, Df(\bar{x})u \rangle + \underbrace{\left\langle u, \frac{1}{\varepsilon}\mathcal{R}(\varepsilon u) \right\rangle}_{|\cdot| \leq \varepsilon\|u\|^3}.$$

We have that

$$|\langle u, Df(\bar{x})u \rangle| \leq -K\|u\|^2$$

for all u and some $K > 0$ (c.f. Arnol's : Differential equation, dynamical systems and an introduction to chaos). The idea is to write $Df(\bar{x})$ into a matrix of the form of $\mathrm{Diag}(\lambda_1, \ldots, \lambda_n) + \mathcal{N}$ where $\mathcal{N}$ is nilpotent, i.e. upper triangular with zero on the diagonal and so that $|(\mathcal{N})_{i<j}| < \varepsilon$.

Therefore $\dot{V}(u) < 0$ for u from any bounded set and small enough ε. □

Warning : This approche doesn't work in the non-autonomous case.

Example

1. Consider $\dot{x} = A(t)x$ with

$$A(t) = \begin{pmatrix} -1 + \frac{3}{2}\cos^2(t) & 1 - \frac{3}{2}\cos(t)\sin(t) \\ -1 - \frac{3}{2}\cos(t)\sin(t) & -1 + \frac{3}{2}\sin^2(t) \end{pmatrix}.$$

Then,

$$\lambda_{1,2}(t) = \frac{1}{4}(-1 \pm i\sqrt{7}),$$

for all t, and thus $\Re(\lambda_i) = -\frac{1}{4} < 0$ for all i. But $\bar{x} = 0$ is not asymptotically stable. Indeed, linear independent solutions are given by

$$v_1 = \begin{pmatrix} -\cos(t) \\ \sin(t) \end{pmatrix} e^{\frac{t}{2}} \quad \text{and} \quad v_{''}(t) = \begin{pmatrix} \sin(t) \\ \cos(t) \end{pmatrix} e^{-t}.$$

Observe that for small $\varepsilon > 0$, $\varepsilon v_1(t)$ is close to $\bar{x} = 0$, but $\|\varepsilon v_1(t)\| \underset{t \to \infty}{\longrightarrow} \infty$.

2. Also, if $y^0 = 0$ is stable for linear system, it doesn't implies that $\bar{x}$ is stable for $\dot{x} = f(x)$. Take

$$\begin{cases} \bar{x} = -y + x(x^2 + y^2) \\ \bar{y} = x + y(x^2 + y^2) \end{cases}.$$

Then, $(\bar{x}, \bar{y}) = (0,0)$ is a stationary solution. The system linearized is given by

$$\begin{cases} \dot{x}^0 = -y^0 \\ \dot{y}^0 = x^0 \end{cases}.$$

C.f. homework  , $Df(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$ and eigenvalues are $\pm i$. The origin is stable for the linearized system (but not asymptotically stable). For the nonlinear, original system, introduce polar coordinate, i.e. $(x, y) = (r\cos(\theta), r\sin(\theta))$. Then the system becomes

$$\begin{cases} \dot{r} = r^3 \\ \dot{\theta} = 1 \end{cases}.$$

Therefore r is increasing for $r(0) > 0$  . So, linearization doesn't help if we have eigenvalues with vanishing real part.

Définition 1.10 (Hyperbolic fixed point)**.**

Let $\bar{x}$ be a stationary point of $\dot{x} = f(x)$. Then $\bar{x}$ is called hyperbolic *if all eigenvalue λ_i of $Df(\bar{x})$ satisfy $\Re(\lambda_i) \neq 0$.*

Let us return to the linearized system $\dot{y}^0_t = Ay^0_t$ with $A = Df(\bar{x})$. So

$$y^0_t = e^{At}y_0.$$

Let λ_j be the eigenvalues of A.

- If v^j is a real eigenvecor with $Av^j = \lambda_j v^j$, then $e^{At}v^j = e^{\lambda t}v^j$. Span$\{v^j\}$ is invariant under the flow $\Phi^t_A(v^j) = e^{At}v^j$.
- If v^j is a complex eigenvector, the same hold for $\Re(v^j)$ and $\Im(v^j)$.

The eigenspaces are invariant under the flow Φ^t_A. We will look with generalized eigenvectors :

Définition 1.11.

1. *A vector $v \neq 0$ is called a generalized eigenvector for A corresponding to λ, if*

$$(A - \lambda I)^k = 0,$$

for a $k \in \mathbb{N}$.

2. *We define the* stable subspace *by*

$$E^s = \operatorname{Span}\{v^1, \ldots, v^{n_s}\},$$

where $v^1, \ldots, v^{n_s}$ are generalized eigenvectors with $\Re(\lambda_i) < 0$ for the associated eigenvalues.

3. *We define the* unstable subspace *by*

$$E^u = \operatorname{Span}\{u^1, \ldots, u^{n_u}\},$$

where $u^1, \ldots, u^{n_u}$ are generalized eigenvectors with $\Re(\lambda_i) > 0$ for the associated eigenvalues.

4. *We denote the* center space *by*

$$E^c = \operatorname{Span}\{w^1, \ldots, w^{n_c}\},$$

where $w^1, \ldots, w^{n_c}$ are generalized eigenvectors with $\Re(\lambda_i) = 0$ for the associated eigenvalues.

Remark 5. $n_s + n_u + n_c = n$.

Solutions in E^c shows the exponential decay (monotonic or oscillatory) and solutions in E^u show the exponential growth. What about E^c ?

1. If there is no multiple eigenvalues $\lambda = 0$, then the solution is constant. If $\lambda_{1,2} = \pm i\beta$ with $\beta \neq 0$, then the oscillation has constant amplitude. C.f. 

2. If there are multiple eigenvalues $\lambda = 0$, then if algebraic and geometric multiplicities differ, solutions in E^c may grow as in the following example.

Example

Let $A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$, and thus $\lambda_{1,2} = 0$ and $v_1 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. All solutions are of the form

$$y_t = e^{At} y_0 = (I + At) y_0 = \begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} y_0.$$

We observe linear growth in the second component.

Let's go back to the non linear system $\dot{x} = f(x)$.

Théorème 1.12 (Hartman-Grobman)**.**

Assume, $\bar{x}$ is a hyperbolic fixed point. Then there exists a homeomorphism h defined on some neighborhood U of $\bar{x}$, which locally maps orbits of the nonlinear flow Φ^t of $\dot{x} = f(x)$ to those of $\Phi^t_{Df(\bar{x})}$ of the linearized system. The homeomorphism preserves the sense of orbits and can be chosen to preserve the parametrization of time.

Remark 6. Under additional conditions (a non-resonance[2] conditions on the eigenvalues), h is a diffeomorphism.

[2] Eigenvalues $\lambda_1, \ldots, \lambda_n$ are called resonant if there exist $i \in \{1, \ldots, n\}$ and $m_i \in \mathbb{N}^*$ with $m_1 + \ldots + m_n \geq 2$ s.t. $\lambda_i = \sum_{j=1}^{n} m_j \lambda_j$.