Dirichlet form

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Contents

Chapter 1

General theory of Dirichlet forms

1.1 Semigroups theory and quadratic forms on Hilbert spaces

Let H be a real Hilbert space with scalar product $\langle \cdot, \cdot \rangle_H$ and norm $\| \cdot \|_H$.

Definition 1.1.

A strongly continuous semi-group of linear operator is a family $(T_t)_{t>0}$ of linear bounded operators $T_t : H \to H$ s.t.

1. $D(T_t) = H$ *for all* $t > 0$,
2. $T_{t+s} = T_t T_s$ *for all* $t, s > 0$
3. $\lim_{t \to 0} \|T_t f - f\|_H = 0$ *for all* $f \in H$.

Moreover, it's contractive if $\|T_t f\|_H \leq \|f\|_H$ for all $f \in H$ and all $t > 0$. It's called symmetric if

$$\langle T_t f, g \rangle_H = \langle f, T_t g \rangle_H ,$$

for all $f, g \in H$ and all $t > 0$.

Example

Take $H = L^2(\mathbb{R}^d)$ and for all $f \in L^2(\mathbb{R}^d)$, define

$$T_t f(x) = \int_{\mathbb{R}^d} p_t(x - y) f(y) \, \mathrm{d}y = (p_t * f)(x),$$

where

$$p_t(x) = \frac{1}{(2\pi t)^{\frac{d}{2}}} e^{-\frac{|x|^2}{2t}}.$$

Then, $(T_t)_{t>0}$ is strongly continuous, symmetric and contractive on $L^2(\mathbb{R}^d)$. It's called the Gaussian Weierstrass semigroup.

Lemma 1.2.

Let $(T_t)_{t>0}$ be a strongly continuous semigroup on H. Then, there exist $M \geq 1$ and $\omega \in \mathbb{R}$ s.t.

$$\|T_t\| \leq M e^{\omega t},$$

for all $t > 0$.

Proof.

Step 1 : Let show that there is $\tau > 0$ s.t. $k = \sup_{0 \le t \le \tau} \|T_t\| < \infty$. Suppose it's not true. Then, there is $(t_n)_{n \ge 0}$ s.t. $t_n \to 0$ and $\|T_{t_n}\| \to \infty$. Then, there is $f \in H$ s.t. $\|T_{t_n} f\| \to \infty$ (by Banach-Steinhauser), which contradict strong continuity.

Step 2 : Given $t \ge 0$, write $t = n\tau + \theta$ with suitable $n \in \mathbb{N}$ and $\theta \in [0, \tau)$. Then,

$$\|T_t\| \le \|T_\tau\|^n \|T_\theta\| \le k^{n+1} \le k(k^{\frac{1}{\tau}})^t.$$

Note that $n \le \frac{t}{\tau}$ and $k \ge \lim_{\varepsilon \to 0^+} \|T_\varepsilon\| = 1$. Consequently, we can use $M = k$ and $\omega = \frac{\log(k)}{\tau}$.

□

Remark

1. Apparently $(T_t)_{t \ge 0}$ is contractive if we can use $M = 1$ and $\omega = 0$.
2. If $(T_t)_{t>0}$ is strongly continuous, then for any $\alpha > 0$ (actually $\alpha \in \mathbb{R}$), $(e^{-\alpha t} T_t)_{t>0}$ is a strongly continuous semigroup.

Example

Look at

$$T_t f(x) = f(x + t), \quad t > 0.$$

It form a strongly continuous semigroup on $L^2(\mathbb{R})$ but it's *not* symmetric. At least for good function; say $f \in \mathcal{C}^1(\mathbb{R}) \cap L^2(\mathbb{R})$; we have

$$\frac{\mathrm{d}^+ f}{\mathrm{d}x}(x) := \lim_{h \to 0^+} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0+} \frac{T_h f(x) - f(x)}{h}.$$

This idea works more generally. The (infinitesimal) generator A of a strongly continuous semigroup $(T_t)_{t>0}$ is defined by

$$\mathcal{D}(A) = \left\{ f \in H \mid \lim_{t \to 0^+} \frac{T_t f - f}{h} \text{ exist in the strong sense in } H \right\},$$

and

$$Af := \lim_{t \to 0^+} \frac{T_t f - f}{t}, \quad f \in \mathcal{D}(A).$$

For example, for the translation semigroup above, $\mathcal{C}^1(\mathbb{R}) \cap L^2(\mathbb{R}) \subset \mathcal{D}(A)$ and $Af = \frac{\mathrm{d}^+}{\mathrm{d}x} f$ for $f \in \mathcal{C}^1(\mathbb{R}) \cap L^2(\mathbb{R})$. In the following, we use Bochner integration (Lebesgue integral on Hilbert spaces).

Lemma 1.3.

Let $(T_t)_{t>0}$ be a strong and continuous semigroup with generator $(A, \mathcal{D}(A))$. Then,

1. $\int_0^t T_s f \, \mathrm{d}s \in \mathcal{D}(A)$ *for all* $f \in H$ *and all* $t > 0$ *and we have that*

$$A\left(\int_0^t T_s f \, \mathrm{d}s\right) = T_t f - f.$$

2. $T_t(\mathcal{D}(A)) \subset \mathcal{D}(A)$ *for all* $t > 0$,

3. *For all* $t > 0$ *and all* $f \in \mathcal{D}(A)$,

$$T_t A f = A T_t f = \frac{\mathrm{d}^+}{\mathrm{d}t} T_t f.$$

In particular, the continuous function $u : [0, \infty) \to H$ *defined by* $u(0) := f$ *and* $u(t) = T_t f$ *solve (uniquely) the Cauchy problem*

$$\begin{cases} \frac{\mathrm{d}^+ u}{\mathrm{d}t}(t) = Au(t) & t > 0 \\ u(0) = f \end{cases}.$$

4. *For all* $t > 0$ *and all* $f \in \mathcal{D}(A)$,

$$T_t f - f = \int_0^t T_s f \, \mathrm{d}s.$$

Remark

If $(T_t)_{t>0}$ is symmetric, then it has better ("regularization") properties, because it's (essentially) an "analytic semigroup". In this case, we have **2.'** $T_t(H) \subset \mathcal{D}(A)$ for all $t > 0$, and the Cauchy problem in **3.** can be solved (uniquely) for any $f \in H$.

Example

The generator of the Gaussian Weierstrass semigroup is $\left(\frac{1}{2}\Delta, H^2(\mathbb{R}^d)\right)$. Let us first check that $\mathcal{D}(A) \supset H^2(\mathbb{R}^d)$. We have that $p_t \in \mathcal{S}(\mathbb{R}^d)$ for all $t > 0$, so $T_t(\mathcal{S}(\mathbb{R}^d)) \subset \mathcal{S}(\mathbb{R}^d)$ for all $t > 0$. Let prove that

$$\lim_{h \to 0} \frac{p_h * f - f}{h} = \frac{1}{2}\Delta f, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

Using Plancherel, the claim is equivalent to

$$\lim_{h \to 0} \frac{1}{(2\pi)^{\frac{d}{2}}} \frac{\hat{p}_h \hat{f} - \hat{f}}{h} = \widehat{\frac{1}{2}\Delta f},$$

where

$$\hat{f}(\xi) = \frac{1}{(1pi)^{\frac{d}{2}}} \int_{\mathbb{R}^d} e^{-i\xi \cdot x} f(x) \, \mathrm{d}x.$$

Since

$$\hat{p}_t(\xi) = e^{-\frac{t\xi^2}{2}} \quad \text{and} \quad \widehat{\Delta f}(\xi) = -\xi^2 \hat{f}(\xi),$$

the claim is equivalent to

$$\lim_{h \to 0} \frac{e^{h\psi} g - g}{h} = \psi g, \quad g \in \mathcal{S}(\mathbb{R}^n), \tag{1.1}$$

where $\psi(\xi) = -\frac{\xi^2}{2}$. To see this last statement, consider

$$\Phi(z) := \frac{e^z - 1}{z} = \sum_{n=2}^{\infty} \frac{z^{n-1}}{n!}.$$

Then,

$$\left\| \frac{e^{\psi h} g - g}{h} - \psi g \right\|_{L^2}^2 = \|\Phi \circ (h\psi) \cdot \psi g\|_{L^2}^2 = \int_{\mathbb{R}^d} \left| \Phi\left(-\frac{h\xi^2}{2}\right) \right|^2 \left| \frac{\xi^2}{2} g(\xi) \right|^2 \mathrm{d}\xi \underset{h \to 0}{\longrightarrow} 0,$$

by dominated convergence theorem. Note that $-1 \leq \Phi(z) \leq 0$ for all $z \leq 0$. Moreover (1.1) remain valid for all $f \in L^2(\mathbb{R}^d)$ s.t. $\psi \hat{f} \in L^2(\mathbb{R}^d)$, that is for all $f \in H^2(\mathbb{R}^d)$. Consequently, $H^2(\mathbb{R}^d) \subset \mathcal{D}(A)$ and

$$Af = \frac{1}{2}\Delta f, \quad f \in H^2(\mathbb{R}^d).$$

Exercices

If $(A, \mathcal{D}(A))$ is the generator of $(T_t)_{t>0}$, find the generator of $(e^{-\alpha t} T_t)_{t>0}$?

Remark

The operators that can occur as generator of strongly continuous semigroup can be caracterized (actually, this work on any Banach space). An interesting example would be the Gaussian-Weierstrass semigroup respectively $-\frac{1}{2}\Delta$ in $\mathcal{C}_0(\mathbb{R}^d)$, the Banach space of continuous function that vanish at infinity.

Theorem 1.4 (Hille-Yoshida)**.**

An operator $(A, \mathcal{D}(A))$ is a generator of a strongly continuous semigroup if and only if the following conditions hold :

1. *$\mathcal{D}(A)$ is dense in H,*
2. *$(A, \mathcal{D}(A))$ is a closed operator,*
3. *There are $\omega \in \mathbb{R}$ and $M \geq 1$ s.t. (ω, ∞) is in the resolvent set of $(A, \mathcal{D}(A))$ and*

$$\|(\lambda - \omega)^n (\lambda - A)^{-n}\| \leq M,$$

for all $\lambda > \omega$ and all $n \in \mathbb{N}$.

*In this case, the corresponding semigroup satisfies $\|T_t\| \leq Me^{\omega t}$ for all $t > 0$ with ω as in **3.**.*

We just verified that generators are closed operators, i.e. operators $(A, \mathcal{D}(A))$ on H for which

$$\Gamma(A) = \{(f, Af) \mid f \in \mathcal{D}(A)\},$$

is a closed subspace of $H \times H$, or equivalently, that $\mathcal{D}(A)$ is a Hilbert space with graph norm

$$\|f\|_{\mathcal{D}(A)} := \|f\| + \|Af\|.$$

Lemma 1.5.

The generator $(A, \mathcal{D}(A))$ of a strongly continuous semigroup is a closed operator.

Proof. Let (f_n) a sequence of $D(A)$ s.t. $f_n \to f$ in H and $Af_n \to g$ in H for some $f, g \in H$. Then, we have $Af = g$. For any $t > 0$,

$$T_t f - f = \lim_{n\to\infty} (T_t f_n - f_n) = \lim_{n\to\infty} \int_0^t T_s A f_n \, \mathrm{d}s = \int_0^t T_s A g \, \mathrm{d}s,$$

and thus

$$\lim_{t\to 0} \frac{T_t f - f}{t} = \lim_{t\to 0} \int_0^t T_s g \, \mathrm{d}s = g.$$

Therefore, $f \in \mathcal{D}(A)$ and $Af = g$. □

Example

Let $A = \frac{1}{2}\Delta$ (Gauss-Wierstrass operator). Then the graph norm in $\mathcal{D}(A)$ is equivalent to $\|\cdot\|_{H^2}$ (c.f. Fourier), and $\mathcal{S}(\mathbb{R}^d) \subset \mathcal{D}(A)$. So, by closedness, we must have $H^2(\mathbb{R}^d) \subset \mathcal{D}(A)$ (by the density of $\mathcal{S}(\mathbb{R}^d)$ in $H^2(\mathbb{R}^d)$). Combining with other implications, $\mathcal{D}(A) = H^2(\mathbb{R}^d)$.

The following notion is very much related to Hille-Yoshida theorem.

Definition 1.6.

A strongly continuous resolvent (with constant $\omega \geq 0$) on H is a family $(G_\alpha)_{\alpha>\omega}$ of linear operators of H s.t.

1. $D(G_\alpha) = H$ *for all* $\alpha > \omega$,
2. $G_\alpha - G_\beta + (\alpha - \beta) G_\alpha G_\beta = 0$ *for all* $\alpha, \beta > \omega$,
3. $\lim_{\alpha\to\infty} \|\alpha G_\alpha f - f\| = 0$ *for all* $f \in H$,
4. *There is* $M \geq 1$ *s.t.*
$$\|(\alpha - \omega) G_\alpha f\| \leq M \|f\|,$$
for all $\alpha > \omega$ *and all* $f \in H$.

Moreover, $(G_\alpha)_{\alpha>\omega}$ is called contractive *if $M = 1$ and $\omega = 0$ and is called* symmetric *if $\langle G_\alpha f, g\rangle = \langle f, G_\alpha g\rangle$ for all $f, g \in H$ and all $\alpha > \omega$.*

Lemma 1.7.

Given a strongly continuous semigroup $(T_t)_{t>0}$, we can define a strongly continuous resolvent $(G_\alpha)_{\alpha>\omega}$ by taking the Laplace transform,

$$G_\alpha f = \int_0^\varepsilon e^{-\alpha t} T_t f \, \mathrm{d}t, \quad f \in H, \tag{1.2}$$

where ω is as in the definition of strong continuity. If $(T_t)_{t>0}$ is symmetric (or contractive), then so is $(G_\alpha)_{\alpha>\omega}$. We call $(G_\alpha)_{\alpha>\omega}$ defined in (1.2) the resolvent of the semigroup $(T_t)_{t>0}$.

Proof. It's easy to show that $(G_\alpha)_{\alpha>\omega}$ defined in (1.2) has all the properties defined in the definition 1.6. □

Example

Gauss-Weierstrass semigroup on $L^2(\mathbb{R}^d)$ is a strong continuous, contractive and symmetric resolvent.

Remark

1. From Hille-Yoshida or **4.** of the definition 1.6, we would expect that G_α should be $(\alpha - A)^{-1}$ if A is the generator of a semigroup. We'll make this precise.

2. Given a strongly continuous resolvent $(G_\alpha)_{\alpha>\omega}$ on H, assume that for some $\alpha > \omega$ we have $G_\alpha u = 0$. Then, $G_\beta u = 0$ for all $\beta > \omega$ by resolvent equation and $u = \lim_{\beta\to\infty} \beta G_\beta u = 0$ by strong continuity. This mean that G_α is invertible. We set

$$\begin{cases} \mathcal{D}(A) := G_\alpha(H), \\ Au := \alpha u - G_\alpha^{-1} u \end{cases}, \quad \alpha > \omega. \tag{1.3}$$

The definition is correct, i.e. doesn't depend on the choice of $\alpha > \omega$. The operator $(A, \mathcal{D}(A))$ is called the generator of the resolvent $(G_\alpha)_{\alpha>\omega}$.

Lemma 1.8.

The generator of a strongly continuous semigroup is equal to the generator of its resolvent.

Proof. Given $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>\omega}$ as annonced. Let A and A' there generators respective. If $f \in D(A')$, thn $f = G_\alpha \varphi$ for some $\varphi \in H$ and

$$\frac{e^{-\alpha t} T_t f - f}{t} = -\frac{1}{t}\int_0^t e^{-\alpha s} T_s \varphi \, \mathrm{d}s \underset{t\to 0}{\longrightarrow} \varphi, \quad \text{in } H,$$

and thus $f \in \mathcal{D}(A)$ and

$$Af = \alpha f - \varphi = A' f.$$

Let $f \in \mathcal{D}(A)$ and set

$$\varphi := \lim_{t\to 0} \frac{e^{-\alpha t} T_t f - f}{t} \quad \text{and} \quad \psi := f - G_\alpha \varphi.$$

Step 1 : We show that $\psi = 0$. If we prove that $G_\beta \psi = 0$, the claim follow because $\psi = \lim_{\beta\to\infty} \beta G_\beta \psi = 0$. So let prove that $G_\beta \psi = 0$. By resolvent equation,

$$G_\beta \psi = G_\beta f - \frac{1}{\alpha - \beta}(G_\beta - G_\alpha)\varphi.$$

Now

$$\begin{aligned}
(G_\beta - G_\alpha)\varphi &= -\lim_{t\to 0} \frac{1}{t}\left[e^{-\alpha t}\int_0^\infty (e^{-\beta s} - e^{-\alpha s}) T_{s+t} f \, \mathrm{d}s - \int_0^\infty (e^{-\beta s} - e^{-\alpha s}) T_s f \, \mathrm{d}s\right] \\
&= -\lim_{t\to 0}\left[e^{-\alpha t}\int_t^\infty (e^{-\beta(u-t)} - e^{-\alpha(u-t)}) T_u f \, \mathrm{d}u - \int_0^\infty (e^{-\beta u} - e^{-\alpha u}) T_u f \, \mathrm{d}u\right] \\
&= -\lim_{t\to 0} \frac{1}{t}\left[\int_0^t e^{-\alpha u} T_u f \, \mathrm{d}u + e^{-(\alpha-\beta)t}\int_t^\infty e^{-\beta u} T_u f \, \mathrm{d}u - \int_0^\infty e^{-\beta u} T_u f \, \mathrm{d}u\right] \\
&= -\lim_{t\to 0} \frac{1}{t}\left[\int_0^t e^{-\alpha u} T_u f \, \mathrm{d}u + (e^{-(\alpha-\beta)t} - 1)\int_t^\infty e^{-\beta u} T_u f \, \mathrm{d}u - \int_0^t e^{-\beta u} T_u f \, \mathrm{d}u\right] \\
&= f + (\alpha - \beta) G_\beta f - f = (\alpha - \beta) G_\beta f.
\end{aligned}$$

Step 2 : By previous step $\psi = 0$. This implies that $f = G_\alpha \varphi \in G_\alpha(H) = D(A')$ and $A' f = \alpha f - \varphi = Af$.

what implies $f = G_\alpha \varphi \in G_\alpha(H) = D(A')$ and $A' f = \alpha f - \varphi = Af$. □

Corollary 1.9.

The generator $(A, \mathcal{D}(A))$ of a strongly continuous semigroup (or strongly continuous resolvent) is densely defined on H, i.e. $\mathcal{D}(A)$ is dense in H.

Proof. $\mathcal{D}(A) = G_\alpha(H)$ for all $\alpha > \omega$, but due to strong continuity, $\mathcal{D}(A)$ must be dense in H, because

$$\lim_{\alpha \to \infty} \| \underbrace{\alpha G_\alpha f}_{\in \mathcal{D}(A)} - f \| = 0.$$

□

- There is also a way back from a resolvent to a semigroup : If $(G_\alpha)_{\alpha > \omega}$ is a strongly continuous resolvent, then
$$T_t f := \lim_{\beta \to \infty} e^{-t\beta} \sum_{n=0}^{\infty} \frac{(t\beta)^n}{n!} (\beta G_\beta)^n f, \quad f \in H,$$
define a strongly continuous semigroup which has resolvent $(G_\alpha)_{\alpha > \omega}$.

- We now look at stucture of generators in the symmetric case. Recall that $(A, \mathcal{D}(A))$ on H is called symmetric if
$$\langle Af, g \rangle = \langle f, Ag \rangle,$$
for all $f, g \in \mathcal{D}(A)$. We could think that this notion generalize the notion of symmetric matrix, but unfortunately it doesn't : we have to find the correct domain.

- Given an unbounded linear operator $(A, \mathcal{D}(A))$ on H with dense domain, we can define the *adjoint operator* $(A^*, \mathcal{D}(A^*))$ where
$$\mathcal{D}(A^*) = \{ f \in H \mid \exists u \in H : \forall g \in \mathcal{D}(A), \langle u, g \rangle = \langle f, Ag \rangle \},$$
and
$$u := A^* f, \quad f \in \mathcal{D}(A^*).$$

Lemma 1.10.

$(A^*, \mathcal{D}(A^*))$ *is a closed operator.*

Proof. Let (f_n) a sequence of $\mathcal{D}(A^*)$ s.t. $f_n \to f$ in H and $A^* f_n \to g$ in H for a certain $f \in H$ and a certain $g \in H$. For all $\varphi \in \mathcal{D}(A)$, we have

$$\langle A\varphi, g \rangle = \lim_{n \to \infty} \langle Af_n, f \rangle = \lim_{n \to \infty} \langle f_n, A^* f \rangle = \langle \varphi, g \rangle,$$

i.e. $\varphi \mapsto \langle A\varphi, f \rangle$ is bounded. Consequently, $f \in \mathcal{D}(A^*)$ and $A^* f = g$. □

- A densely defined operator $(A, \mathcal{D}(A))$ on H is called self adjoint if $\mathcal{D}(A^*) = \mathcal{D}(A)$ and $A^* f = Af$ on $\mathcal{D}(A)$. There is a "hidden closure process" in the definition of the adjoint, dictates the "correct domain".

- By Lemma 1.10, we know that a self adjoint operator is closed. Also, it's symmetric. A symmetric operator is self adjoint if $\mathcal{D}(A) = H$, but it's not true in general.

Example

$A := i\frac{\mathrm{d}}{\mathrm{d}x}$, $H = L^2(-1,1)$ and

$$\mathcal{D}(A) = \{f \in \mathcal{C}^1([-1,1]) \mid f(-1) = f(1) = 0\}.$$

Then, A is densely defined and symmetric. But it's not closed and thus not self adjoint. Indeed, consider

$$f_n(x) = \left(x^2 + \frac{1}{n}\right)^{\frac{1}{2}}, \quad f(x) = |x| \quad \text{and} \quad g(x) = \operatorname{sgn}(x),$$

with the convention that $\operatorname{sgn}(0) = 0$. Then, $f_n \to f$ in $L^2(-1,1)$ (the convergence is actually uniform on $[-1,1]$), $Af_n \to ig$ in L^2 but $f \notin \mathcal{D}(A)$.

- The notion of self-adjoint operator is the "domain-wise correct" generalization of the notion of symmetric matrix.

Remark

For a densely operator $(A, \mathcal{D}(A))$, we have

1. A symmetric if and only if $A \subset A^*$ (i.e. $\mathcal{D}(A) \subset \mathcal{D}(A^*)$ and $Af = A^*f$ for all $f \in \mathcal{D}(A)$). In this case, $A^{**} = (A^*)^*$ can be defined as the smallest closed extension of A, and we have $A \subset A^{**} \subset A^*$.
2. A is closed and symmetric, i.e. $A = A^{**} \subset A^*$.
3. A is self adjoint if and only if $A = A^{**} = A^*$. This is the case if and only if the adjoint A^* is a symmetric operator.

Example

Consider again $A = i\frac{\mathrm{d}}{\mathrm{d}x}$ on $L^2(-1,1)$ with a domain that makes it closed, namely

$$\mathcal{D}(A) = \{f \in L^2(-1,1) \mid f \text{ is equal to an absolute continuous function } \tilde{f} \text{ a.e. on } [-1,1] \text{ and s.t. } \tilde{f}(-1) = \tilde{f}(1) = 0\}.$$

Then A is symmetric an

$$\mathcal{D}(A^*) = \left\{f \in L^2(-1,1) \mid t \text{ is equal to an absolute continuous function } \tilde{f} \text{ a.e. on } [-1,1] \text{ s.t. } \int_{-1}^{1} |\tilde{f}|^2 < \infty\right\},$$

and $A^* = i\frac{\mathrm{d}}{\mathrm{d}x}$. Then, A^* is not symmetric because $e^{-x} \in \mathcal{D}(A^*)$ but $A^*e^{-x} = -ie^{-x}$, i.e. $-i$ is a non zero imaginary imaginary part eigenvalue, which is impossible for a symmetric operator.

Definition 1.11.

A symmetric operator $(A, \mathcal{D}(A))$ *is called* semi-bounded *if there is* $C \geq 0$ *s.t.*

$$\langle -Af, f\rangle \geq -C\|f\|^2,$$

for all $f \in \mathcal{D}(A)$. *If* $\langle -Af, f\rangle \geq 0$, *then it's called non-positive definite.*

Lemma 1.12.

The generator of a symmetric strongly continuous resolvent (or symmetric strongly continuous semi-group) on H *is a semi-bounded self-adjoint operator if the resolvent (or semi-group) is contractive. Then, its generator is non positive definite.*

Proof. Since G_α is symmetric and defined on a all H, it's therefore self-adjoint. By the spectral theorem (see below) also G_α^{-1} is self adjoint, and then so is A. Set $\varphi(\alpha) := \langle f, G_\alpha f\rangle$, where $f \in \mathcal{D}(A)$ and $\alpha > \omega$. By the resolvent equation,

$$G_{\alpha+\varepsilon} f - G_\alpha = \varepsilon G_{\alpha+\varepsilon} G_\alpha f,$$

so that

$$\varphi'(\alpha) = -\langle G_\alpha f, G_\alpha f\rangle \leq 0.$$

By **4.** of the definition of a resolvent,

$$|\varphi(\alpha)| \leq \frac{M}{\alpha}\|f\|^2 \underset{\alpha\to\infty}{\longrightarrow} 0.$$

Therefore $\varphi(\alpha) \geq 0$ for all $\alpha > \omega$ and thus, G_α is non-negative definite. Then, for all $f \in \mathcal{D}(A)$, ($\mathcal{D}(A) = G_\alpha(H)$ for all $\alpha > \omega$),

$$\langle (\omega - \alpha) f, f\rangle = \lim_{\alpha\to\omega} \langle -Af + \alpha f, f\rangle = \lim_{\alpha\to\omega} \left\langle G_\alpha^{-1} f, f\right\rangle = 0.$$

So,

$$\langle -Af, f\rangle \geq -\omega\|f\|^2,$$

for all $f \in \mathcal{D}(A)$. □

We now gives two version of the spectral theorem (without proof).

Theorem 1.13 (Spectral decomposition)**.**

Let $-A : H \to H$ be self adjoint with domain $\mathcal{D}(A)$. Then, there is a spectral measure (spectral family) $(E_\lambda)_{\lambda\in\mathbb{R}}$ s.t.

$$-\langle Af, g\rangle = \int_{\mathbb{R}} \lambda \,\mathrm{d}\langle E_\lambda f, g\rangle, \quad f \in \mathcal{D}(A), g \in H,$$

and given a measurable function $\varphi : \mathbb{R} \to \mathbb{R}$ and setting

$$D\big(\varphi(-1)\big) = \left\{ f \in H \mid \int_{\mathbb{R}} |\varphi(\lambda)| \,\mathrm{d}\langle E_\lambda f, f\rangle < \infty \right\},$$

then $\varphi(-A)$ defined by

$$\langle \varphi(-A) f, f\rangle = \int_{\mathbb{R}} \varphi(\lambda) \,\mathrm{d}\langle E_\lambda f, g\rangle, \quad f \in \mathcal{D}(A), g \in H,$$

define a self adjoint operator $\Big(\varphi(-A), D\big(\varphi(-A)\big)\Big)$. For $\varphi = id$, we recover

$$\mathcal{D}(A) = \left\{ f \in H \mid \int_{\mathbb{R}} \lambda^2 \,\mathrm{d}\langle E_\lambda f, f\rangle < \infty \right\}.$$

Example

1. In many application, one encounters the situation that

$$\mathcal{D}(A) = \left\{ f \in H \mid \sum_{i=0}^{\infty} \lambda_i^2 |\langle \varphi_i, f\rangle|^2 < \infty \right\},$$

with a sequence of real numbers $(\lambda_i)_{i\in\mathbb{R}}$ and a complete orthonormal system $(\varphi_i)_{i\in\mathbb{N}}$ in H, and

$$Af = \sum_{i=0}^{\infty} \lambda_i \langle \varphi_i, f\rangle \varphi_i, \quad f \in \mathcal{D}(A).$$

This is for instance the case if "A has a pure point spectrum" to the λ_i, one refers as eigenvalues of A and to the φ_i as eigenfunctions (e.g. second order operators on bounded domain).

2. Given a non positive definite self-adjoint operator $(A, \mathcal{D}(A))$, we can use the spectral theorem to define fractional power of $-A$ by looking at $\varphi(\lambda) = \lambda^\alpha$, $\alpha \in \mathbb{R}$

As this version of the spectral theorem shows : all self-adjoint operator are multiplication operator up to a unitary transformation.

Theorem 1.14 (Spectral theorem).

Let $-A : H \to H$ be self-adjoint with domain $\mathcal{D}(A)$. Then, there exist a measure space $(\Omega, \mathcal{F}, \mu)$, a measurable function $\Phi : \Omega \to \mathbb{R}$ and a unitary operator $U : H \to L^2(\Omega, \mu)$ s.t. $f \in \mathcal{D}(A)$ if and only if $\Phi \cdot Uf \in L^2(\Omega, \mu)$. Moreover, defining

$$D(M_\Phi) = \{\varphi \in L^2(\Omega, \mu) \mid \Phi \cdot Uf \in L^2(\Omega, \mu)\},$$

and

$$M_\Phi(g) := \Phi \cdot g, \quad g \in D(M_\Phi),$$

the operator $(M_\Phi, D(M_\Phi))$ is self-adjoint on $L^2(\Omega, \mu)$ and

$$M_\Phi(g) = U(-A)U^* g, \quad g \in D(M_\Phi).$$

The spectral theorem is also a way to construct (symmetric) semigroup and resolvent from self adjoint operator.

Lemma 1.15.

Let $(A, \mathcal{D}(A))$ semi-bounded self-adjoint operator on H (with constant ω, i.e. $\langle -Af, f\rangle \geq -\omega \|f\|^2$),

1. *Setting $T_t := e^{tA}$ (i.e. $\varphi(\lambda) = e^{\lambda t}$), $t > 0$ and $G_\alpha := (\alpha - A)^{-1}$ (i.e. $\varphi(\lambda) = \frac{1}{\alpha+\lambda}$), $\alpha > \omega$, we obtain a symmetric strongly continuous semigroup $(T_t)_{t>0}$ and a symmetric strongly continuous resolvent $(G_\alpha)_{\alpha>\omega}$ on H.*
2. *The generator of $(T_t)_{t>0}$ is A and $(T_t)_{t>0}$ is the only semigroup with this generator. Similarly for the resolvent.*

Proof. **1.** Follows directly from spectral theorem : for $\varphi, \psi : [-c, \infty) \to \mathbb{R}$ continuous, we have

$$\langle \varphi(-1)u, \psi(-A)v\rangle = \int_{[-c,\infty)} \varphi(\lambda)\psi(\lambda)\, \mathrm{d}\langle E_\lambda u, v\rangle,$$

for all $u \in D(\varphi(-A)$ and $v \in D(\psi(-A))$. We apply this to $\lambda \mapsto e^{-t\lambda}$ and $\lambda \mapsto \frac{1}{\alpha+\lambda}$. For instance, writing in symbolic notation ("operator calculus"), $\alpha G_\alpha^{-1} = \int_{[-c,\infty)} \left(\frac{\alpha}{\alpha+\lambda} - 1\right) \mathrm{d}E_\lambda$ implies

$$\langle \alpha G_\alpha u - u, \alpha G_\alpha u - u\rangle = \int_{[-c,\infty)} \left(\frac{\alpha}{\alpha+\lambda} - 1\right)^2 \mathrm{d}\langle E_\lambda u, u\rangle \underset{\alpha\to\infty}{\longrightarrow} 0,$$

for any $u \in H$. Tis show the strong continuity of $(G_\alpha)_{\alpha>c}$.

2. For any $f \in H$ and $\alpha > c$, we have

$$\int_{[-c,\infty)} \lambda^2\, \mathrm{d}\langle E_\lambda G_\alpha f, G_\alpha f\rangle = \int_{[-c,\infty)} \frac{\lambda^2}{(\alpha+\lambda)^2}\, \mathrm{d}\langle E_\lambda f, f\rangle < \infty,$$

i.e. $G_\alpha(H) \subset \mathcal{D}(A)$. Since $(\alpha - A)G_\alpha g = f$, $f \in H$ and $G_\alpha(\alpha - A)f = f$, $f \in \mathcal{D}(A)$ (also to be seen via spectral theorem), we see that A is the generator of $(G_\alpha)_{\alpha>c}$. Let $(G'_\alpha)_{\alpha>c}$ be a strongly continuous resolvent generated by A. Given $f \in H$, consider

$$w = G_\alpha f - G'_\alpha f \implies (\alpha - A)w,$$

for all $\alpha > c$. Since $c - A$ is non negative definite and $\alpha > c$, we get $w = 0$. This shows $G'_\alpha = G_\alpha$ for $\alpha > c$. The uniqueness of the semigroup follow from the right continuity of $t \mapsto \langle T_t f, g\rangle$ and the uniqueness theorem for Laplace transformation.

□

After having looked at semigroups, resolvents and generators, we add another perpective :

Definition 1.16.

A densely defined bilinear form $(Q, \mathcal{D}(Q))$ on H is a bilinear map $Q : \mathcal{D}(Q) \times \mathcal{D}(Q) \to \mathbb{R}$ where $\mathcal{D}(Q)$ is a dense subspace of H. If $Q(f,g) = Q(g,f)$ for all $f, g \in \mathcal{D}(Q)$, we say that $(Q, \mathcal{D}(Q))$ is symmetric. It's called semi-bounded if there is $C \geq 0$ s.t.

$$Q(f,f) \geq -C\|f\|^2,$$

for all $f \in \mathcal{D}(Q)$ and non-negative definite if this is true for $C = 0$. A semi-bounded form is closed if $\mathcal{D}(Q)$ is a Hilbert space with norm

$$\|f\|_{Q,\alpha} := \sqrt{Q_\alpha(f,f)}, \quad f \in \mathcal{D}(Q),$$

for some $\alpha > c$ where

$$Q_\alpha(f,g) := Q(f,g) + \alpha \langle f, g\rangle, \quad f, g \in \mathcal{D}(Q).$$

Remark

1. If this hold for one $\alpha > c$, then it holds for all $\alpha > c$ and the Hilbert norms $\|\cdot\|_{Q,\alpha}$, $\alpha > c$ are equivalents norms.
2. Some author refer to a densely defined symmetric closed form as "closed quadratic form".

Any semi-bounded self-adjoint operator generates such a form :

Lemma 1.17.

Let $(A, \mathcal{D}(A))$ be a semi-bounded self adjoint operator on H with spectral representation

$$\langle -Af, g\rangle = \int_{[-c,\infty)} \lambda \, \mathrm{d}\langle E_\lambda f, g\rangle, \quad f \in \mathcal{D}(A), g \in H,$$

and

$$\mathcal{D}(A) = \left\{ f \in H \mid \int_{[-c,\infty)} \lambda^2 \, \mathrm{d}\langle E_\lambda f, f\rangle < \infty \right\}.$$

Then

$$Q(f,g) := \int_{[-c,\infty)} \lambda \, \mathrm{d}\langle E_\lambda f, g\rangle, \quad f, g \in \mathcal{D}(Q),$$

$$\mathcal{D}(Q) = \left\{ f \in H \mid \int_{[-c,\infty)} \lambda \, \mathrm{d}\langle E_\lambda f, f\rangle < \infty \right\},$$

defined a densely defined symmetric semi-bounded and closed form $(Q, \mathcal{D}(Q))$ with

$$Q(f,f) \geq -C\|f\|^2, \quad f \in \mathcal{D}(Q).$$

If A is non-positive definite (i.e. $-A$ is positive definite), then $(Q, \mathcal{D}(Q))$ is non-negative definite and $\mathcal{D}(Q) = \mathcal{D}(\sqrt{-A})$,

$$Q(f,g) = \left\langle \sqrt{-A}f, \sqrt{-A}g \right\rangle.$$

Proof. It suffice to consider the case that A is non-positive definite (otherwise look at $A - \alpha$). We need to verify density of $\mathcal{D}(Q)$ and closedness. But $\sqrt{-A}$ is a (non-negative defnite) self-adjoint operator. In particular, densely defined and closed. Clearly, $\mathcal{D}(Q)$ is dense, and the closedness follows from the closedness of $\sqrt{-A}$. □

Corollary 1.18.

For the strongly continuous resolvent $(G_\alpha)_{\alpha>c}$ generated by $(A, \mathcal{D}(A))$, we have $G_\alpha(H) \subset \mathcal{D}(Q)$, $\alpha > c$, and

$$Q_\alpha(G_\alpha f, g) = \langle f, g \rangle\,, \quad f \in H, g \in \mathcal{D}(Q). \tag{1.4}$$

If A is non-positive definite, then this holds for all $\alpha > 0$. The form $(Q, \mathcal{D}(Q))$ is characterized by

$$\mathcal{D}(A) \subset \mathcal{D}(Q) \quad \text{and} \quad Q(f, g) = \langle -Af, g \rangle\,, \quad f \in \mathcal{D}(A), g \in \mathcal{D}(Q). \tag{1.5}$$

Remark

The formula (1.5) is called "abstract Gauss-Green formula".

The power of closed quadratic form is that, unlike for operators, it cannot happen that they are symmetric and closed "but not self-adjoint". In other words, the following holds :

Lemma 1.19.

Given a closed quadratic form $(Q, \mathcal{D}(Q))$ on H, there exists a unique self-adjoint operator $(A, \mathcal{D}(A))$ s.t. (1.5) holds. This operator is semi-bounded, it is non-positive definite if and only if $(Q, \mathcal{D}(Q))$ is non-negative definite.

Proof. For any $\alpha > c$ and any $u \in H$, there is a unique element $G_\alpha u \in \mathcal{D}(A)$ s.t.

$$Q_\alpha(G_\alpha u, v) = \langle u, v \rangle\,, \quad v \in \mathcal{D}(Q). \tag{1.6}$$

By Riesz representation theorem. One can see that $(G_\alpha)_{\alpha>c}$, where G_α is the operator $u \mapsto G_\alpha u$, $u \in H$, is a symmetric strongly continuous resolvent :

$$(\alpha - c)\|G_\alpha u\|^2 \underset{\text{semibd.}}{\leq} Q_\alpha(G_\alpha u, G_\alpha u) = \langle u, G_\alpha u \rangle \leq \|u\|\|G_\alpha u\|,$$

which implies the bound **4.** in the definition. To see strong continuity, we can use boundedness **4.** and density of $\mathcal{D}(Q)$ in H to restrict attention to the question wether

$$\beta G_\beta u \to u \quad \text{as } \beta \to \infty,$$

for $u \in \mathcal{D}(Q)$. This follow from

$$\begin{aligned}(\beta - c)\|\beta G_\beta u - u\|^2 &\leq Q_\beta(\beta G_\beta u - u, \beta G_\beta u - u)\\ &= \beta \langle \beta G_\beta u, u \rangle - 2\beta \langle u, u \rangle + Q(u, u) \leq Q_c(u, u),\end{aligned}$$

which implies convergence and therefore **3.** in the definition of the resolvent. To see the resolvent equation **2.**, suppose $\alpha, \beta > c$. Then

$$\begin{aligned}Q_\alpha(G_\beta u - (\alpha - \beta)G_\alpha G_\beta u, v) &= Q_\beta(G_\beta u, v) + (\alpha - \beta) \langle G_\beta u, v \rangle - (\alpha - \beta) \langle G_\beta u, v \rangle\\ &= \langle u, v \rangle = Q_\alpha(G_\alpha u, v),\end{aligned}$$

where $u \in H$ and $v \in \mathcal{D}(Q)$. Let $(A, \mathcal{D}(A))$ be the generator $(G_\alpha)_{\alpha>c}$. Since A is semi-bounded and self-adjoint, it generates a closed quadratic form $(Q', D(Q'))$ satisfying (1.5). We claim that $Q' = Q$. The formula (1.4) for Q' implies that $G_\alpha(H) \subset D(Q')$ and

$$Q'_\alpha(G_\alpha u, G_\alpha v) = \langle u, G_\alpha v \rangle = Q_\alpha(G_\alpha u, G_\alpha v), \quad u, v \in H.$$

Therefore $Q' = Q$ on $G_\alpha(H) \times G_\alpha(H)$. Since by (1.4) and (1.6), $G_\alpha(H)$ is dense (because $\mathcal{D}(A)$ is dense) both in $\mathcal{D}(Q)$ qnd $D(Q')$, we get $Q' = Q$. For a given Q, the self-adjoint operator satisfying (1.5) is unique, because the resolvent satisfy (1.4) and it determines $(G_\alpha)_{\alpha>c}$ uniquely and therefore also A. □

As a consequence, we have the following

Theorem 1.20.

There is a one-to-one correspondance between the family of non-negative definite closed quadratic forms and the family of non-positive definite self-adjoint operator, it is given by (1.5).

$(A, \mathcal{D}(A))$ is a non-positive self-adjoint operator if and only if $(Q, \mathcal{D}(Q))$ is a non-negative definite closed quadratic form, where is equivalnce is given by the spectral theorem. Spectral theorem also provide approximation formulas.

Lemma 1.21.

Let $(Q, \mathcal{D}(Q))$ a non-negative definite closed quadratic form and let $(A, \mathcal{D}(A))$ be it's generator. Let $(T_t)_{t>0}$ and $(G_\alpha)_{\alpha>0}$ be the associated semigroup and resolvent (strongly continuous, symmetric and contractive). Then,

1. *For all $t > 0$, $T_t(H) \subset \mathcal{D}(Q)$*[a] *and*

$$Q(T_t u, T_t u) \leq \frac{1}{2t}\big(\langle u, u\rangle - \langle T_t u, T_t u\rangle\big) \leq Q(u, u), \quad u \in \mathcal{D}(Q).$$

2. *For all $\alpha > 0$, $G_\alpha(H) \subset \mathcal{D}(Q)$, and*

$$Q_\alpha(G_\alpha u, v) = \langle u, v\rangle, \quad u \in H, v \in \mathcal{D}(Q).$$

3. *Given $u \in \mathcal{D}(Q)$, we have*

$$T_t u \underset{t\to 0^+}{\longrightarrow} u,$$

$$\frac{1}{t}\big(G_1 u - e^{-t} G_1 T_t u\big) = \frac{1}{t}\big(G_1 u - e^{-t} T_t G_1 u\big) \underset{t\to 0^+}{\longrightarrow} u,$$

and

$$\alpha G_\alpha u \underset{\alpha\to\infty}{\longrightarrow} u,$$

all strongly in $\mathcal{D}(Q)$.

[a] Think for example $T_t(L^2) \subset H^1$.

Proof. Via spectral theorem. □

Another, practically important type of approximation is as follow : define symmetric bilinear form on H by

$$Q^{(t)}(u, v) := \frac{1}{t}\langle u - T_t u, v\rangle, \quad u, v \in H, t > 0,$$

and

$$Q^{(\beta)}(u, v) := \beta\langle u - \beta G_\beta u, v\rangle, \quad u, v \in H, \beta > 0.$$

Lemma 1.22.

Let Q, $-A$, T_t, G_α, $Q^{(t)}$ and $Q^{(\beta)}$ as above.

1. *For any $u \in H$, $Q^{(t)}(u,u)$ is non-decreasing as $t \to 0^+$, and*

$$\mathcal{D}(Q) = \left\{ u \in H \mid \lim_{\to 0^+} Q(u,u) < \infty \right\},$$

$$Q(u,v) = \lim_{t \to 0^+} Q^{(t)}(u,v), \quad u, v \in \mathcal{D}(Q).$$

2. *For any $u \in H$, $Q^{(\beta)}(u,v)$ is non decreasing as $\beta \nearrow \infty$ and*

$$\mathcal{D}(Q) = \left\{ u \in H \mid \lim_{\beta \to \infty} Q(u,u) < \infty \right\},$$

$$Q(u,v) = \lim_{\beta \to \infty} Q^{(\beta)}(u,v), \quad u, v \in \mathcal{D}(Q).$$

Proof. Via spectral theorem. □

1.2 Markov semigroup and Dirichlet forms

From now on, assume that $(X, \mathcal{X}, m)$ is a $\sigma-$finite measure space. Write $L^p := L^p(m) = L^p(X, \mathcal{X}, m)$, $0 < p \leq \infty$. We consider the case $H = L^2 = L^2(m)$. So, "we start to do probability".

Definition 2.23.

A bounded symmetric operator P on L^2 is called Markovian *(or just* Markov*) if P is symmetric and for all $f \in L^2$ s.t. $0 \leq f \leq 1$, $m-$a.e. we have $0 \leq Pf \leq 1$, $m-$a.e.*

Remark

1. In particular, Markov operators are positivity preserving, i.e. $Pf \geq 0$, $m-$a.e. whenever $f \in L^2$ is s.t. $f \geq 0$, $m-$a.e.
2. In some text, the above operators are called *sub-Markovian*, and only operators satisfying $P\mathbf{1}_X = \mathbf{1}_X$ are called Markovian. But we'll use the more general manner of speaking.
3. Passing via L^1, we see that any Markov operator is contractive in L^2.

Proposition 2.24.

Suppose $f \in L^1 \cap L^2$. Then $Pf \in L^1$ and $\|Pf\|_{L^1} \leq \|f\|_{L^1}$.

Proof. Let $f \in L^1 \cap L^2$ and $\varphi \in L^1 \cap L^\infty$, $\gamma \geq 0$. Then

$$\int \varphi |Pf| \,\mathrm{d}m \leq \int \varphi P(|f|) \,\mathrm{d}m \leq \int P(\varphi)|f| \,\mathrm{d}m \leq \|\varphi\|_{L^\infty} \int |f| \,\mathrm{d}m.$$

Now, take a sequence $(\varphi_n)_n$ of function $0 \leq \varphi_n \leq 1$ for all n, with $\varphi_n \nearrow \mathbf{1}_X$ pointwise. □

We can extend the restriction of P to $L^1 \cap L^2$ to a Markov contraction on L^1. We denote this extension $P^{(1)}$.

Proposition 2.25.

Let $f \in L^2$. Then $(Pf)^2 \leq P^{(1)} f^2$.

Proof. It suffices to prove the property for $f \in L^1 \cap L^\infty$ (by boundedness of P and density). Let $A \in \mathcal{X}$ with $m(A) < \infty$. Since for all $\lambda \in \mathbb{R}$ we have

$$P(f + \lambda \mathbf{1}_A)^2 \geq 0, \quad m - \text{a.e.},$$

we obtain

$$[P(f\mathbf{1}_A)]^2 \leq P(f^2)P(\mathbf{1}_A) \leq P(f), \quad m - \text{a.e.}$$

By $\sigma-$finiteness, we can find $A_n \in \mathcal{X}$ s.t. $\mathbf{1}_{A_n} \nearrow \mathbf{1}_X$ pointwise. Dominated convergence theorem gives the result. □

Definition 2.26.

A function $F : \mathbb{R}^n \to \mathbb{R}$ with

$$|F(x) - F(y)| \leq \sum_{i=1}^n |x_i - y_i|, \quad x, y \in \mathbb{R}^n,$$

and $F(0) = 0$ is called normal contraction.

Given a Markov operator P on L^2, we consider the quadratic form

$$\mathcal{E}_p(f) := \langle f - Pf, f \rangle_{L^2}, \quad f \in L^2.$$

Proposition 2.27 (Implication of Markovity)**.**

Let $F : \mathbb{R}^p \to \mathbb{R}$ be a normal contraction, P a Markov operator and $f_1, \dots, f_p \in L^2$. Then, writing $f = (f_1, \dots, f_p)$, we have

$$(\mathcal{E}_p(F \circ f))^{\frac{1}{2}} \leq \sum_{i=1}^p (\mathcal{E}_p(f_i))^{\frac{1}{2}}.$$

To prove the proposition, we use an elementary lemma :

Lemma 2.28.

Let $A_1, \dots, A_n \in \mathcal{X}$, pairwise disjoints of finite measure, and consider

$$\varphi = \sum_{i=1}^n \alpha_i \mathbf{1}_{A_i},$$

with $\alpha_i \in \mathbb{R}$. Let

$$\lambda_i = m(A_i), \quad \varepsilon_{ij} = \left\langle \mathbf{1}_{A_i}, P\mathbf{1}_{A_j} \right\rangle_{L^2} \quad \text{and} \quad \mu_j = \lambda_j - \sum_{i=1}^n a_{ij}.$$

Then $\mu_j \geq 0$ for all j and

$$\mathcal{E}_p(\varphi) = \sum_{i=1}^n \mu_i \alpha_i^2 + \frac{1}{2} \sum_{1 \leq i,j \leq n} \varepsilon_{ij} (\alpha_i - \alpha_j)^2.$$

Proof. The equality

$$\mu_j = \left\langle \mathbf{1}_{A_j}, \mathbf{1} - P\left(\sum_{i=1}^{n} \mathbf{1}_{A_i}\right)\right\rangle_{L^2},$$

hold. Since the A_i's are pairwise disjoints and P is Markov, the μ_j are non-negative. One have

$$\begin{aligned}
\mathcal{E}_p(\varphi) &= \sum_{i,j} \langle \alpha_i(\mathbf{1}_{A_i} - P\mathbf{1}_{A_i}), \alpha_j \mathbf{1}_{A_j} \rangle \\
&= \sum_i \alpha_i^2 \mu_i - \sum_{i,j} \varepsilon_{ij} \alpha_i \alpha_j \\
&= \sum_i \alpha_i^2 \mu_i + \sum_i \alpha_i^2 \sum_j \varepsilon_{ij} - \sum_{i,j} \alpha_i \alpha_j \varepsilon_{ij}.
\end{aligned}$$

□