Analysis of differential and integro-differential equations

Prof. S. Herr

October 21, 2018

Contents

1 Laplace equation **2**
1.1 Harmonic functions . . . 2
1.2 Newtonian potential . . . 5

Chapter 1

Laplace equation

1.1 Harmonic functions

The Laplace operator is defined as

$$\Delta = \sum_{j=1}^{n} \partial_j^2.$$

Définition 1.1.

Let $U \subset \mathbb{R}^n$ open and $u \in \mathcal{C}^2(U)$.

1. *u is subharmonic if $\Delta u \geq 0$ in U,*
2. *u is superharmonic if $\Delta u \leq 0$ in U,*
3. *u is harmonic if $\Delta u = 0$.*

Examples

1. Affine and linear functions,
2. $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = x^2 - y^2$,
3. $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $v(x, y) = \log(x^2 + y^2)$.

Théorème 1.2 (Mean value property (MVP))**.**

Let $U \subset \mathbb{R}^n$ open, $\overline{\mathcal{B}_r(x)} \subset U$ and $u \in \mathcal{C}^2(U)$.

1. *If u is harmonic in U, then*

$$u(x) = \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma = \fint_{\mathcal{B}_r(x)} u,$$

where $\fint_A u = \frac{1}{|A|} \int_A u$, for a measurable set A.

2. *If u is subharmonic (resp. superharmonic) in U, then*

$$u(x) \underset{(\geq)}{\leq} \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma \quad \text{and} \quad u(x) \underset{(\geq)}{\leq} \fint_{\mathcal{B}_r(x)} u.$$

We set $\omega_n := |\mathcal{B}_1(0)|$ and recall that

$$|\mathcal{B}_r(x)| = r^n\omega_n \quad \text{and} \quad |\partial\mathcal{B}_r(x)| = nr^{n-1}\omega_n.$$

Proof. It suffice to prove **2.** when u is subharmonic. Let $0 < \rho \leq r$. By Divergence,

$$\int_{\partial\mathcal{B}_\rho(x)} \nabla u \cdot \eta \,\mathrm{d}\sigma = \int_{\mathcal{B}_\rho} \operatorname{div}(\nabla u) \geq 0.$$

Define

$$f(\rho) = \fint_{\partial\mathcal{B}_\rho(x)} u \,\mathrm{d}\sigma.$$

We have

$$\begin{aligned}
0 &\leq \int_{\partial\mathcal{B}_\rho(x)} u \,\mathrm{d}\sigma \\
&= \rho^{n-1} \int_{\partial\mathcal{B}_1(0)} \nabla u(x+\rho t)\cdot t \,\mathrm{d}\sigma(t) \\
&= \rho^{n-1}\frac{\partial}{\partial\rho} \int_{\partial\mathcal{B}_1(0)} u(x+\rho t)\,\mathrm{d}\sigma(t) \\
&= \rho^{n-1} n\omega_n \frac{\partial}{\partial\rho} \fint_{\partial\mathcal{B}_\rho(0)} u \\
&= \rho^{n-1} n\omega_n f'(\rho),
\end{aligned}$$

and thus f is increasing. Since

$$\lim_{\rho\to 0^+} f(\rho) = u(x),$$

we get $u(x) \leq f(\rho)$ for all $0 < \rho \leq r$, and thus, the first inequality follow. For the second one,

$$r^n\omega_n u(x) = \int_0^r n\omega_n\rho^{n-1}u(x)\,\mathrm{d}\rho \leq \int_0^r \int_{\partial\mathcal{B}_\rho(x)} u\,\mathrm{d}\sigma = \int_{\mathcal{B}_r(x)} u,$$

and thus the second inequality follow. □

Théorème 1.3.

Let $U \subset \mathbb{R}^n$ a domain (i.e. open and connected).

1. *If $u \in \mathcal{C}^2(U)$ is harmonic and there is $x \in U$ s.t.*

$$u(x) = \sup_U u \quad \text{or} \quad u(x) = \inf_U u,$$

then u is constant.

2. *If $u \in \mathcal{C}^2(U)$ is sub harmonic (resp. superharmonic) and there is x s.t. $u(x) = \sup_U u$ (resp. $u(x) = \inf_U u$), then u is constant.*

Proof. It suffice to prove **2.** when u is subharmonic. Let $s := \sup_U u$ and set

$$U_s := \{x \in U \mid u(x) = s\}.$$

Since u is continuous, the set U_s is closed. It's also open since if $x \in U_s$ and $\mathcal{B}_r(x) \subset U$, by MVP applied to $u - s$,

$$0 = u(z) - s \leq \int_{\mathcal{B}_r(x)} \underbrace{(u-s)}_{\leq 0} \leq 0.$$

Therefore

$$\int_{\mathcal{B}_r(x)} (u-s) = 0,$$

and thus $u(y) = s$ for all $y \in \mathcal{B}_r(x)$. Then $\mathcal{B}_r(x) \subset U$ and the claim follow. □

Remark 1. **1.** The proof only use MVP.

2. If U is open and bounded, the minimum and the maximum of harmonic function (resp. maximum of subharmonic/ minimum of superharmonic) are taken on the boundary.

Corolaire 1.4.

Let $U \subset \mathbb{R}^n$ is open, bounded and $u, v \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$.

1. *If $\Delta u = \Delta v$ in U and $u = v$ on ∂U, then $u = v$.*
2. *If u is subharmonic (resp. superharmonic), v harmonic and $u = v$ on ∂U, then $u \leq v$ (resp. $u \geq v$) in U.*

Proof. Apply the previous remark to $w := u - v$. □

Théorème 1.5 (Harnack's inequality)**.**

Let $U \subset \mathbb{R}^n$ be a domain and V relatively compact[a] sub-domain (i.e. $V \subset U$ and V connected). Then, there is a constant $C > 0$ s.t. for all harmonic function $u \in \mathcal{C}^2(U)$,

$$\sup_V u \leq C \inf_V u.$$

[a] i.e. $\overline{V}$ is compact in U

Proof.

Step 1 : Let $0 < \delta \leq \frac{1}{4}\mathrm{dist}(\overline{V}, \partial U)$ and $x, y \in V$ s.t. $|x - y| \leq \delta$. Then

$$\mathcal{B}_\delta(y) \subset \mathcal{B}_{2\delta}(x) \subset U.$$

By the MVP,

$$u(x) = \fint_{\mathcal{B}_{2\delta}(x)} u \underset{u \geq 0}{\geq} \frac{|\mathcal{B}_\delta(x)|}{|\mathcal{B}_{2\delta}(x)|} \int_{\mathcal{B}_\delta(x)} u = 2^{-n} u(y).$$

By symmetry, we get

$$2^{-n} u(y) \leq u(x) \leq 2^n u(y),$$

for all $|x - y| \leq \delta$.

Step 2 : By compactness of $\overline{V}$, there is a finite covering of ball $B_1, \ldots, B_N$ of radius $\frac{\delta}{2}$ and centers $p_1, \ldots, p_N$. If $x, y \in V$, there are $B_{j_1}, \ldots, B_{j_m}$ of those ball, $m \leq N$ s.t. $x \in B_{j_1}$, $B_{j_k} \cap B_{j_{k+1}} \neq \varnothing$ and $y \in B_{j_m}$. By step 1,

$$u(x) \leq 2^n u(p_{j_1}) \leq \ldots \leq 2^{n(N+1)} u(y).$$

The claim follow with $C = 2^{n(N+1)}$.Peut-être plutôt $2^{n(m+1)}$?

□

1.2 Newtonian potential

Définition 1.6.

The Newtonain potential $\Gamma : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ *is defined as*

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log |x| & n = 2 \\ -\frac{1}{n(n-2)\omega_n} |x|^{2-n} & n \geq 3. \end{cases}$$

Remark 2.

$$\partial_i \Gamma(x) = \frac{1}{n\omega_n} \frac{x_i}{|x|^n} \quad \text{and} \quad \frac{1}{n\omega_n} \cdot \frac{\delta_{ij}|x|^2 - n x_i x_j}{|x|^{n+1}}.$$

Then

$$\Delta\Gamma(x) = \frac{1}{n\omega_n |x|^{n+2}} \sum_{j=1}^{n} (|x|^2 - n x_j^2) = 0.$$

Also

$$|\partial_i \Gamma(x)| \leq \frac{1}{n\omega_n} \cdot \frac{1}{|x|^{n-1}} \quad \text{and} \quad |\partial_j \partial_i \Gamma(x)| \leq \frac{1}{\omega_n} \cdot \frac{1}{|x|^n}.$$

Théorème 1.7.

Let $f \in \mathcal{C}_c^1(\mathbb{R}^n)$ *and*

$$u(x) = (\Gamma * f)(x) := \int_{\mathbb{R}^n} \Gamma(x-y) f(y) \, \mathrm{d}y = \int_{\mathbb{R}^n} \Gamma(y) f(x-y) \, \mathrm{d}y.$$

Proof. The functions

$$x \mapsto \Gamma(x-y) \quad \text{and} \quad y \mapsto \Gamma(x-y),$$

are harmonics on $\mathbb{R}^n \setminus \{x\}$ and $\mathbb{R}^n \setminus \{y\}$ respectively and integrable on compacts sets.

$$\begin{aligned} \partial_i u(x) &= \int_{\mathbb{R}^n} \Gamma(y) \partial_i f(x-y) \, \mathrm{d}y \\ &= \int_{\mathbb{R}^n} \Gamma(x-y) \partial_i f(y) \, \mathrm{d}y \\ &= \underbrace{\int_{\mathcal{B}_\varepsilon(x)} \Gamma(x-y) \partial_i f(y) \, \mathrm{d}y}_{=:A} + \underbrace{\int_{\mathcal{B}_\varepsilon(x)^c} \Gamma(x-y) \partial_i f(y) \, \mathrm{d}y}_{=:B}. \end{aligned}$$

By divergence theorem[1]

$$B = -\int_{\partial \mathcal{B}_\varepsilon(x)} \Gamma(x-y) f(y) \eta_i(y) \, \mathrm{d}\sigma(y) + \int_{\mathcal{B}_\varepsilon(x)^c} \partial_i \Gamma(x-y) f(y) \, \mathrm{d}y,$$

where $\eta(y) = \frac{y-x}{|y-x|}$ the exterior normal unit vector. We have

$$\begin{aligned} |A| &\leq \|\partial_i f\|_{L^\infty(\mathbb{R}^n)} \int_{\mathcal{B}_\varepsilon(0)} \leq \begin{cases} \omega_n \int_0^\varepsilon r^{n-1} r^{n-2} \frac{1}{n\omega_n(n-2)} \, \mathrm{d}r & n \geq 3 \\ \omega_2 \int_0^\varepsilon \frac{r \log(r)}{2\pi} \, \mathrm{d}r & n = 2 \end{cases} \\ &\leq C \|\partial_i f\|_{L^\infty(\mathbb{R}^n)} \varepsilon \underset{\varepsilon \to 0^+}{\longrightarrow} 0. \end{aligned}$$

[1] Recall that $\mathrm{div}(fF) = f\mathrm{div}(F) + F \cdot \nabla f$, and use divergence theorem with $F(y) = (0, \ldots, 0, \partial_i f(y), 0, \ldots, 0)$ where $\partial_i f(y)$ is at the i^{th} position and $f(y) = \Gamma(x-y)$.

$$|B| \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon^{n-1}\varepsilon^{2-n} & n \geq 3 \\ \varepsilon|\log(\varepsilon)| & n = 2 \end{cases} \underset{\varepsilon \to 0^+}{\longrightarrow} 0.$$

Therefore

$$\partial_i u(x) = \int_{\mathbb{R}^n} \partial_i \Gamma(x-y) f(y) \, \mathrm{d}y.$$

As above

$$\partial_j \partial_i u(x) = \int_{\mathcal{B}_\varepsilon(x)} \partial_i \Gamma(x-y) \partial_j f(y) \, \mathrm{d}y + \int_{\mathcal{B}_\varepsilon(x)^c} \partial_j \partial_i \Gamma(x-y) f(y) \, \mathrm{d}y.$$

Therefore,

$$\begin{aligned} \Delta u(x) &= -\sum_{j=1}^n \int_{\partial\mathcal{B}_\varepsilon(x)} \partial_j \Gamma(x-y) \frac{y_j - x_j}{|y-x|} f(y) \, \mathrm{d}\sigma(y) + \int_{\mathcal{B}_\varepsilon(x)^c} \underbrace{\Delta\Gamma(x-y)}_{=0} f(y) \, \mathrm{d}y \\ &= \int_{\partial\mathcal{B}_\varepsilon(x)} \underbrace{\nabla\Gamma(x-y) \cdot \frac{x-y}{|x-y|}}_{=\frac{1}{n\omega_n}|x-y|^{-1}} f(y) \, \mathrm{d}y \\ &= \fint_{\partial\mathcal{B}_\varepsilon(x)} f(y) \, \mathrm{d}y \underset{\varepsilon \to 0^+}{\longrightarrow} f(x). \end{aligned}$$

□

Remark 3. The condition $f \in \mathcal{C}_c^1(\mathbb{R}^n)$ is not optimal, however $f \in \mathcal{C}_c(\mathbb{R}^n)$ is not enough.

Proposition 1.8 (Green Formula)**.**

Let $U \subset \mathbb{R}^n$ be a open and bounded s.t. $\partial U \in \mathcal{C}^1$, $u \in \mathcal{C}^2(\overline{U})$ (i.e. $\partial_i u$ and $\partial_{ij} u$ are extendable on ∂U by continuity). For $x \in U$,

$$u(x) = \int_{\partial U} \left(u(y) \frac{\partial \Gamma}{\partial \eta}(y-x) - \Gamma(y-x) \frac{\partial u}{\partial \eta} \right) \mathrm{d}\sigma(y) - \int_U \Gamma(y-x) \Delta u(y) \, \mathrm{d}y,$$

where η is the exterior unit normal vector field w.r.t. U and $\frac{\partial}{\partial \eta} := \eta \cdot \nabla$.

Proof. Let $x \in U$, $\varepsilon > 0$, $\overline{\mathcal{B}_\varepsilon(x)} \subset U$ and $U_\varepsilon := U \setminus \mathcal{B}_\varepsilon(x)$. Using divergence formula, one gets [2]

$$\underbrace{\int_{U_\varepsilon} \Gamma(y-x) \Delta u(y) \, \mathrm{d}y}_{\underset{\varepsilon \to 0^+}{\longrightarrow} \int_U \Gamma(y-x)\Delta u(y)\,\mathrm{d}y} = \int_{\partial U_\varepsilon} \left(\Gamma(y-x) \frac{\partial u}{\partial \eta}(y) - \frac{\partial \Gamma}{\partial \eta}(y-x) u(y) \right) \mathrm{d}\sigma(y).$$

By theorem 1.7,

$$\int_{\partial\mathcal{B}_\varepsilon(x)} \Gamma(y-x) \frac{\partial u}{\partial \eta}(y) \, \mathrm{d}\sigma(y) \underset{\varepsilon \to 0^+}{\longrightarrow} 0.$$

For $y \in \partial\mathcal{B}_\varepsilon(x)$, $\eta(y) = -\frac{y-x}{|y-x|}$, hence

$$-\int_{\partial\mathcal{B}_\varepsilon(x)} \frac{\partial \Gamma}{\partial \eta}(y-x) u(y) \, \mathrm{d}\sigma(y) \underset{\varepsilon \to 0^+}{\longrightarrow} u(x).$$

Why don't we only consider $\partial\mathcal{B}_\varepsilon(x)$ and not ∂U in ∂U_ε ? □

[2] We use $\operatorname{div}(u\nabla v - v\nabla u)$ for well chosen functions.

Définition 1.9.

Let $U \subset \mathbb{R}^n$ be open and bounded. For $x \in U$, let $h^x \in \mathcal{C}^2(U) \cap \mathcal{C}^1(\overline{U})$ be the solution (if it exist) of

$$\begin{cases} \Delta h^x = 0 & \text{in } U \\ h^x = -\Gamma(\cdot - x) & \text{on } \partial U. \end{cases}$$

The Green function of U is defined by

$$\begin{aligned} G : U \times U \setminus \{(u,u) \mid u \in U\} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto -\Gamma(y-x) - h^x(y). \end{aligned}$$

Remark 4. Existence is unclear but unicity comes from maximum principle.

Corolaire 1.10 (Green representation formula)**.**

Let $U \subset \mathbb{R}^n$ open and bounded, $\partial U \in \mathcal{C}^1$ and G the Green function of U. If $u \in \mathcal{C}^2(\overline{U})$ is a solution to

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

then for $x \in U$,

$$u(x) = \int_U G(x,y) f(y) \,\mathrm{d}y - \int_{\partial U} \underbrace{\frac{\partial G}{\partial \eta}(x,y)}_{=\eta(y)\cdot \nabla_y G(x,y)} \, g(y) \,\mathrm{d}\sigma(y).$$

Proof. Apply Green formula to h^x. ☐

Remark 5.

1. $K = -\frac{\partial G}{\partial \eta}$ is called *Poisson kernel for U*.
2. One can prove that $G \geq 0$ and $G(x,y) = G(y,x)$.

Proposition 1.11.

Let $n \geq 2$. Then, the Green function of $U = \mathcal{B}_1(0)$ is given by

$$G(x,y) = \begin{cases} \Gamma(|x|) \left(y - \frac{x}{|x|^2} \right) - \Gamma(y - x) & x \neq 0 \\ \Gamma(1) - \Gamma(y) & x = 0, \end{cases}$$

for all $(x,y) \in U \times U \setminus \{(u,u) \mid u \in U\}$, and Poisson kernel is given by

$$-\frac{\partial G}{\partial \eta}(x,y) = \frac{1 - |x|^2}{n\omega_n |x-y|^n}.$$

Proof.

Step 1 : We have that

$$h^x(y) = \begin{cases} -\Gamma\big(|x|(y - \bar{x}\big) & x \neq 0 \\ -\Gamma(1) & x = 0, \end{cases}$$

where $\bar{x} = \frac{x}{|x|^2}$. Indeed, we have that Γ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Moreover if $x \in \mathcal{B}_1(0)$, then $\bar{x} \notin \mathcal{B}_1(0)$. For $y \in \partial\mathcal{B}_1(0)$, we have that

$$h^x(y) = -\Gamma(y - x),$$

since for $x \neq 0$,

$$\big||x|(y - \bar{x})^2\big| = |x|^2 \left(|y|^2 - \frac{2xy}{|x|^2} + \frac{1}{|x|^2}\right) = |x - y|^2.$$

Step 2 : For the Poisson kernel, we have that

$$\nabla\Gamma(y - x) = \frac{1}{n\omega_n} \frac{y - x}{|y - x|}.$$

Also,

$$\nabla_y h^x(y) = -\nabla_y \Gamma\Big(|x|(y - \bar{x})\Big)|x| = \frac{1}{n\omega_n}|x|^2 \frac{y - x}{|y - x|}.$$

Since $\eta(y) = y$,

$$\begin{aligned}
-\frac{\partial G}{\partial \eta}(x, y) &= -\frac{\partial \Gamma}{\partial \eta}(y - x) - \frac{\partial h^x}{\partial \eta}(y) \\
&= -\frac{1}{n\omega_n}\left(\frac{(y - x)\cdot y}{|y - x|^n} - \frac{|x|^2 y \cdot y - x \cdot y}{|y - x|^2}\right).
\end{aligned}$$

□

By translation and scaling, we get :

Théorème 1.12 (Poisson formula)**.**

Let $r > 0$, $x_0 \in \mathbb{R}^n$ and $g \in \mathcal{C}\left(\partial\mathcal{B}_r(x_0)\right)$. Then,

$$u(x) = \begin{cases} \frac{r^2 - |x - x_0|^2}{n\omega_n r} \int_{\partial\mathcal{B}_r(x_0)} \frac{g(y)}{|x - y|^n}\, \mathrm{d}\sigma(y) & x \in \mathcal{B}_r(x_0) \\ g(x) & x \in \partial\mathcal{B}_r(x_0), \end{cases}$$

is a $\mathcal{C}^\infty(\mathcal{B}_r(x_0)) \cap \mathcal{C}\left(\overline{\mathcal{B}_r(x_0)}\right)$ function solving the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{B}_r(x_0) \\ u = g & \text{on } \partial\mathcal{B}_r(x_0). \end{cases}$$

Proof. Prove that u is really a solution of the Dirichlet problem. □