Analysis of differential and integro-differential equations

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Contents

Chapter 1

Laplace equation

1.1 Harmonic functions

The Laplace operator is defined as

$$\Delta = \sum_{j=1}^{n} \partial_j^2.$$

Définition 1.1.

Let $U \subset \mathbb{R}^n$ open and $u \in \mathcal{C}^2(U)$.

1. *u is subharmonic if $\Delta u \geq 0$ in U,*
2. *u is superharmonic if $\Delta u \leq 0$ in U,*
3. *u is harmonic if $\Delta u = 0$.*

Examples

1. Affine and linear functions,
2. $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = x^2 - y^2$,
3. $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $v(x, y) = \log(x^2 + y^2)$.

Théorème 1.2 (Mean value property (MVP))**.**

Let $U \subset \mathbb{R}^n$ open, $\overline{\mathcal{B}_r(x)} \subset U$ and $u \in \mathcal{C}^2(U)$.

1. *If u is harmonic in U, then*

$$u(x) = \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma = \fint_{\mathcal{B}_r(x)} u,$$

where $\fint_A u = \frac{1}{|A|} \int_A u$, for a measurable set A.

2. *If u is subharmonic (resp. superharmonic) in U, then*

$$u(x) \underset{(\geq)}{\leq} \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma \quad \text{and} \quad u(x) \underset{(\geq)}{\leq} \fint_{\mathcal{B}_r(x)} u.$$

We set $\omega_n := |\mathcal{B}_1(0)|$ and recall that

$$|\mathcal{B}_r(x)| = r^n\omega_n \quad \text{and} \quad |\partial\mathcal{B}_r(x)| = nr^{n-1}\omega_n.$$

Proof. It suffice to prove **2.** when u is subharmonic. Let $0 < \rho \leq r$. By Divergence,

$$\int\limits_{\partial\mathcal{B}_\rho(x)} \nabla u \cdot \eta \,\mathrm{d}\sigma = \int\limits_{\mathcal{B}_\rho} \operatorname{div}(\nabla u) \geq 0.$$

Define

$$f(\rho) = \fint\limits_{\partial\mathcal{B}_\rho(x)} u \,\mathrm{d}\sigma.$$

$$\begin{aligned}
0 &\leq \int\limits_{\partial\mathcal{B}_\rho(x)} u \,\mathrm{d}\sigma \\
&= \rho^{n-1} \int\limits_{\partial\mathcal{B}_1(0)} \nabla u(x+\rho t)\cdot t \,\mathrm{d}\sigma(t) \\
&= \rho^{n-1}\frac{\partial}{\partial\rho} \int\limits_{\partial\mathcal{B}_1(0)} u(x+\rho t)\,\mathrm{d}\sigma(t) \\
&= \rho^{n-1} n\omega_n \frac{\partial}{\partial\rho} \fint\limits_{\partial\mathcal{B}_\rho(0)} u \\
&= \rho^{n-1} n\omega_n f'(\rho),
\end{aligned}$$

and thus f is increasing. Since

$$\lim_{\rho\to 0^+} f(\rho) = u(x),$$

we get $u(x) \leq f(\rho)$ for all $0 < \rho \leq r$, and thus, the first inequality follow. For the second one,

$$r^n\omega_n u(x) = \int_0^r n\omega_n\rho^{n-1}u(x)\,\mathrm{d}\rho \leq \int_0^r \int\limits_{\partial\mathcal{B}_\rho(x)} u\,\mathrm{d}\sigma = \int\limits_{\mathcal{B}_r(x)} u,$$

and thus the second inequality follow. □

Théorème 1.3.

Let $U \subset \mathbb{R}^n$ a domain (i.e. open and connected).

1. *If $u \in \mathcal{C}^2(U)$ is harmonic and there is $x \in U$ s.t.*

$$u(x) = \sup_U u \quad \text{or} \quad u(x) = \inf_U u,$$

   then u is constant.

2. *If $u \in \mathcal{C}^2(U)$ is sub harmonic (resp. superharmonic) and there is x s.t. $u(x) = \sup_U u$ (resp. $u(x) = \inf_U u$), then u is constant.*

Proof. It suffice to prove **2.** when u is subharmonic. Let $s := \sup_U u$ and set

$$U_s := \{x \in U \mid u(x) = s\}.$$

Since u is continuous, the set U_s is closed. It's also open since if $x \in U_s$ and $\mathcal{B}_r(x) \subset U$, by MVP applied to $u - s$,

$$0 = u(z) - s \leq \int_{\mathcal{B}_r(x)} \underbrace{(u-s)}_{\leq 0} \leq 0.$$

Therefore

$$\int_{\mathcal{B}_r(x)} (u-s) = 0,$$

and thus $u(y) = s$ for all $y \in \mathcal{B}_r(x)$. Then $\mathcal{B}_r(x) \subset U$ and the claim follow. □

Remark 1. **1.** The proof only use MVP.

2. If U is open and bounded, the minimum and the maximum of harmonic function (resp. maximum of subharmonic/ minimum of superharmonic) are taken on the boundary.

Corolaire 1.4.

Let $U \subset \mathbb{R}^n$ is open, bounded and $u, v \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$.

1. *If $\Delta u = \Delta v$ in U and $u = v$ on ∂U, then $u = v$.*
2. *If u is subharmonic (resp. superharmonic), v harmonic and $u = v$ on ∂U, then $u \leq v$ (resp. $u \geq v$) in U.*

Proof. Apply the previous remark to $w := u - v$. □

Théorème 1.5 (Harnack's inequality)**.**

Let $U \subset \mathbb{R}^n$ be a domain and V relatively compact[a] sub-domain (i.e. $V \subset U$ and V connected). Then, there is a constant $C > 0$ s.t. for all harmonic function $u \in \mathcal{C}^2(U)$,

$$\sup_V u \leq C \inf_V u.$$

[a] i.e. $\overline{V}$ is compact in U

Proof.

Step 1 : Let $0 < \delta \leq \frac{1}{4}\text{dist}(\overline{V}, \partial U)$ and $x, y \in V$ s.t. $|x - y| \leq \delta$. Then

$$\mathcal{B}_\delta(y) \subset \mathcal{B}_{2\delta}(x) \subset U.$$

By the MVP,

$$u(x) = \fint_{\mathcal{B}_{2\delta}(x)} u \underset{u \geq 0}{\geq} \frac{|\mathcal{B}_\delta(x)|}{|\mathcal{B}_{2\delta}(x)|} \int_{\mathcal{B}_\delta(x)} u = 2^{-n} u(y).$$

By symmetry, we get

$$2^{-n} u(y) \leq u(x) \leq 2^n u(y),$$

for all $|x - y| \leq \delta$.

Step 2 : By compactness of $\overline{V}$, there is a finite covering of ball $B_1, \ldots, B_N$ of radius $\frac{\delta}{2}$ and centers $p_1, \ldots, p_N$. If $x, y \in V$, there are $B_{j_1}, \ldots, B_{j_m}$ of those ball, $m \leq N$ s.t. $x \in B_{j_1}$, $B_{j_k} \cap B_{j_{k+1}} \neq \varnothing$ and $y \in B_{j_m}$. By step 1,

$$u(x) \leq 2^n u(p_{j_1}) \leq \ldots \leq 2^{n(N+1)} u(y).$$

The claim follow with $C = 2^{n(N+1)}$. Peut-être plutôt $2^{n(m+1)}$?

□

1.2 Newtonian potential

Définition 1.6.

The Newtonain potential $\Gamma : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ *is defined as*

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi} \log|x| & n = 2 \\ -\frac{1}{n(n-2)\omega_n}|x|^{2-n} & n \geq 3. \end{cases}$$

Remark 2.

$$\partial_i \Gamma(x) = \frac{1}{n\omega_n} \frac{x_i}{|x|^n} \quad \text{and} \quad \frac{1}{n\omega_n} \cdot \frac{\delta_{ij}|x|^2 - nx_i x_j}{|x|^{n+1}}.$$

Then

$$\Delta\Gamma(x) = \frac{1}{n\omega_n |x|^{n+2}} \sum_{j=1}^{n} (|x|^2 - nx_j^2) = 0.$$

Also

$$|\partial_i \Gamma(x)| \leq \frac{1}{n\omega_n} \cdot \frac{1}{|x|^{n-1}} \quad \text{and} \quad |\partial_j \partial_i \Gamma(x)| \leq \frac{1}{\omega_n} \cdot \frac{1}{|x|^n}.$$