Analysis of differential and integro-differential equations

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October 25, 2018

Contents

Chapter 1

Laplace equation

1.1 Harmonic functions

The Laplace operator is defined as

$$\Delta = \sum_{j=1}^{n} \partial_j^2.$$

Définition 1.1.

Let $U \subset \mathbb{R}^n$ open and $u \in \mathcal{C}^2(U)$.

1. *u is subharmonic if $\Delta u \geq 0$ in U,*
2. *u is superharmonic if $\Delta u \leq 0$ in U,*
3. *u is harmonic if $\Delta u = 0$.*

Examples

1. Affine and linear functions,
2. $u : \mathbb{R}^2 \to \mathbb{R}$ defined by $u(x, y) = x^2 - y^2$,
3. $v : \mathbb{R}^2 \setminus \{0\} \to \mathbb{R}$ defined by $v(x, y) = \log(x^2 + y^2)$.

Théorème 1.2 (Mean value property (MVP))**.**

Let $U \subset \mathbb{R}^n$ open, $\overline{\mathcal{B}_r(x)} \subset U$ and $u \in \mathcal{C}^2(U)$.

1. *If u is harmonic in U, then*

$$u(x) = \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma = \fint_{\mathcal{B}_r(x)} u,$$

where $\fint_A u = \frac{1}{|A|} \int_A u$, for a measurable set A.

2. *If u is subharmonic (resp. superharmonic) in U, then*

$$u(x) \underset{(\geq)}{\leq} \fint_{\partial \mathcal{B}_r(x)} u \, \mathrm{d}\sigma \quad \textit{and} \quad u(x) \underset{(\geq)}{\leq} \fint_{\mathcal{B}_r(x)} u.$$

We set $\omega_n := |\mathcal{B}_1(0)|$ and recall that

$$|\mathcal{B}_r(x)| = r^n\omega_n \quad \text{and} \quad |\partial\mathcal{B}_r(x)| = nr^{n-1}\omega_n.$$

Proof. It suffice to prove **2.** when u is subharmonic. Let $0 < \rho \leq r$. By Divergence,

$$\int_{\partial\mathcal{B}_\rho(x)} \nabla u\cdot\eta \,\mathrm{d}\sigma = \int_{\mathcal{B}_\rho} \operatorname{div}(\nabla u) \geq 0.$$

Define

$$f(\rho) = \fint_{\partial\mathcal{B}_\rho(x)} u\,\mathrm{d}\sigma.$$

We have

$$\begin{aligned}
0 &\leq \int_{\partial\mathcal{B}_\rho(x)} u\,\mathrm{d}\sigma \\
&= \rho^{n-1}\int_{\partial\mathcal{B}_1(0)} \nabla u(x+\rho t)\cdot t\,\mathrm{d}\sigma(t) \\
&= \rho^{n-1}\frac{\partial}{\partial\rho}\int_{\partial\mathcal{B}_1(0)} u(x+\rho t)\,\mathrm{d}\sigma(t) \\
&= \rho^{n-1}n\omega_n\frac{\partial}{\partial\rho}\fint_{\partial\mathcal{B}_\rho(0)} u \\
&= \rho^{n-1}n\omega_n f'(\rho),
\end{aligned}$$

and thus f is increasing. Since

$$\lim_{\rho\to 0^+} f(\rho) = u(x),$$

we get $u(x)\leq f(\rho)$ for all $0<\rho\leq r$, and thus, the first inequality follow. For the second one,

$$r^n\omega_n u(x) = \int_0^r n\omega_n\rho^{n-1}u(x)\,\mathrm{d}\rho \leq \int_0^r\int_{\partial\mathcal{B}_\rho(x)} u\,\mathrm{d}\sigma = \int_{\mathcal{B}_r(x)} u,$$

and thus the second inequality follow. ☐

Théorème 1.3.

Let $U\subset\mathbb{R}^n$ a domain (i.e. open and connected).

1. *If $u\in\mathcal{C}^2(U)$ is harmonic and there is $x\in U$ s.t.*

$$u(x) = \sup_U u \quad \textit{or} \quad u(x) = \inf_U u,$$

then u is constant.

2. *If $u\in\mathcal{C}^2(U)$ is sub harmonic (resp. superharmonic) and there is x s.t. $u(x) = \sup_U u$ (resp. $u(x) = \inf_U u$), then u is constant.*

Proof. It suffice to prove **2.** when u is subharmonic. Let $s := \sup_U u$ and set

$$U_s := \{x\in U \mid u(x) = s\}.$$

Since u is continuous, the set U_s is closed. It's also open since if $x \in U_s$ and $\mathcal{B}_r(x) \subset U$, by MVP applied to $u - s$,

$$0 = u(z) - s \leq \int_{\mathcal{B}_r(x)} \underbrace{(u-s)}_{\leq 0} \leq 0.$$

Therefore

$$\int_{\mathcal{B}_r(x)} (u-s) = 0,$$

and thus $u(y) = s$ for all $y \in \mathcal{B}_r(x)$. Then $\mathcal{B}_r(x) \subset U$ and the claim follow. ☐

Remark 1. **1.** The proof only use MVP.

2. If U is open and bounded, the minimum and the maximum of harmonic function (resp. maximum of subharmonic/ minimum of superharmonic) are taken on the boundary.

Corolaire 1.4.

Let $U \subset \mathbb{R}^n$ is open, bounded and $u, v \in \mathcal{C}^2(U) \cap \mathcal{C}(\overline{U})$.

1. *If $\Delta u = \Delta v$ in U and $u = v$ on ∂U, then $u = v$.*
2. *If u is subharmonic (resp. superharmonic), v harmonic and $u = v$ on ∂U, then $u \leq v$ (resp. $u \geq v$) in U.*

Proof. Apply the previous remark to $w := u - v$. ☐

Théorème 1.5 (Harnack's inequality)**.**

Let $U \subset \mathbb{R}^n$ be a domain and $V \subset\subset U$ [a] sub-domain (i.e. $V \subset U$ and V connected). Then, there is a constant $C > 0$ s.t. for all harmonic function $u \in \mathcal{C}^2(U)$,

$$\sup_V u \leq C \inf_V u.$$

[a] i.e. V is relatively compact : $\overline{V}$ is compact in U

Proof.

Step 1 : Let $0 < \delta \leq \frac{1}{4}\text{dist}(\overline{V}, \partial U)$ and $x, y \in V$ s.t. $|x - y| \leq \delta$. Then

$$\mathcal{B}_\delta(y) \subset \mathcal{B}_{2\delta}(x) \subset U.$$

By the MVP,

$$u(x) = \fint_{\mathcal{B}_{2\delta}(x)} u \underset{u \geq 0}{\geq} \frac{|\mathcal{B}_\delta(x)|}{|\mathcal{B}_{2\delta}(x)|} \int_{\mathcal{B}_\delta(x)} u = 2^{-n} u(y).$$

By symmetry, we get

$$2^{-n} u(y) \leq u(x) \leq 2^n u(y),$$

for all $|x - y| \leq \delta$.

Step 2 : By compactness of $\overline{V}$, there is a finite covering of ball $B_1, \ldots, B_N$ of radius $\frac{\delta}{2}$ and centers $p_1, \ldots, p_N$. If $x, y \in V$, there are $B_{j_1}, \ldots, B_{j_m}$ of those ball, $m \leq N$ s.t. $x \in B_{j_1}$, $B_{j_k} \cap B_{j_{k+1}} \neq \varnothing$ and $y \in B_{j_m}$. By step 1,

$$u(x) \leq 2^n u(p_{j_1}) \leq \ldots \leq 2^{n(N+1)} u(y).$$

The claim follow with $C = 2^{n(N+1)}$.Peut-être plutôt $2^{n(m+1)}$?

☐

1.2 Newtonian potential

Définition 1.6.

The Newtonain potential $\Gamma : \mathbb{R}^n \setminus \{0\} \longrightarrow \mathbb{R}$ *is defined as*

$$\Gamma(x) = \begin{cases} \frac{1}{2\pi}\log|x| & n = 2 \\ -\frac{1}{n(n-2)\omega_n}|x|^{2-n} & n \geq 3. \end{cases}$$

Remark 2.

$$\partial_i \Gamma(x) = \frac{1}{n\omega_n}\frac{x_i}{|x|^n} \quad \text{and} \quad \frac{1}{n\omega_n}\cdot\frac{\delta_{ij}|x|^2 - nx_ix_j}{|x|^{n+1}}.$$

Then

$$\Delta\Gamma(x) = \frac{1}{n\omega_n|x|^{n+2}}\sum_{j=1}^{n}(|x|^2 - nx_j^2) = 0.$$

Also

$$|\partial_i\Gamma(x)| \leq \frac{1}{n\omega_n}\cdot\frac{1}{|x|^{n-1}} \quad \text{and} \quad |\partial_j\partial_i\Gamma(x)| \leq \frac{1}{\omega_n}\cdot\frac{1}{|x|^n}.$$

Théorème 1.7.

Let $f \in \mathcal{C}_c^1(\mathbb{R}^n)$ *and*

$$u(x) = (\Gamma * f)(x) := \int_{\mathbb{R}^n}\Gamma(x-y)f(y)\,\mathrm{d}y = \int_{\mathbb{R}^n}\Gamma(y)f(x-y)\,\mathrm{d}y.$$

Proof. The functions

$$x \mapsto \Gamma(x-y) \quad \text{and} \quad y \mapsto \Gamma(x-y),$$

are harmonics on $\mathbb{R}^n \setminus \{x\}$ and $\mathbb{R}^n \setminus \{y\}$ respectively and integrable on compacts sets.

$$\begin{aligned} \partial_i u(x) &= \int_{\mathbb{R}^n}\Gamma(y)\partial_i f(x-y)\,\mathrm{d}y \\ &= \int_{\mathbb{R}^n}\Gamma(x-y)\partial_i f(y)\,\mathrm{d}y \\ &= \underbrace{\int_{\mathcal{B}_\varepsilon(x)}\Gamma(x-y)\partial_i f(y)\,\mathrm{d}y}_{=:A} + \underbrace{\int_{\mathcal{B}_\varepsilon(x)^c}\Gamma(x-y)\partial_i f(y)\,\mathrm{d}y}_{=:B}. \end{aligned}$$

By divergence theorem[1]

$$B = -\int_{\partial\mathcal{B}_\varepsilon(x)}\Gamma(x-y)f(y)\eta_i(y)\,\mathrm{d}\sigma(y) + \int_{\mathcal{B}_\varepsilon(x)^c}\partial_i\Gamma(x-y)f(y)\,\mathrm{d}y,$$

where $\eta(y) = \frac{y-x}{|y-x|}$ the exterior normal unit vector. We have

$$\begin{aligned} |A| &\leq \|\partial_i f\|_{L^\infty(\mathbb{R}^n)}\int_{\mathcal{B}_\varepsilon(0)} \leq \begin{cases} \omega_n\int_0^\varepsilon r^{n-1}r^{n-2}\frac{1}{n\omega_n(n-2)}\,\mathrm{d}r & n \geq 3 \\ \omega_2\int_0^\varepsilon \frac{r\log(r)}{2\pi}\,\mathrm{d}r & n = 2 \end{cases} \\ &\leq C\|\partial_i f\|_{L^\infty(\mathbb{R}^n)}\varepsilon \underset{\varepsilon\to 0^+}{\longrightarrow} 0. \end{aligned}$$

[1] Recall that $\mathrm{div}(fF) = f\mathrm{div}(F) + F\cdot\nabla f$, and use divergence theorem with $F(y) = (0,\ldots,0,\partial_i f(y),0,\ldots,0)$ where $\partial_i f(y)$ is at the i^{th} position and $f(y) = \Gamma(x-y)$.

$$|B| \leq C\|f\|_{L^\infty(\mathbb{R}^n)} \begin{cases} \varepsilon^{n-1}\varepsilon^{2-n} & n \geq 3 \\ \varepsilon|\log(\varepsilon)| & n = 2 \end{cases} \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

Therefore

$$\partial_i u(x) = \int_{\mathbb{R}^n} \partial_i \Gamma(x-y) f(y)\,\mathrm{d}y.$$

As above

$$\partial_j \partial_i u(x) = \int_{\mathcal{B}_\varepsilon(x)} \partial_i \Gamma(x-y) \partial_j f(y)\,\mathrm{d}y + \int_{\mathcal{B}_\varepsilon(x)^c} \partial_j \partial_i \Gamma(x-y) f(y)\,\mathrm{d}y.$$

Therefore,

$$\begin{aligned}
\Delta u(x) &= -\sum_{j=1}^n \int_{\partial\mathcal{B}_\varepsilon(x)} \partial_j \Gamma(x-y) \frac{y_j - x_j}{|y-x|} f(y)\,\mathrm{d}\sigma(y) + \int_{\mathcal{B}_\varepsilon(x)^c} \underbrace{\Delta\Gamma(x-y)}_{=0} f(y)\,\mathrm{d}y \\
&= \int_{\partial\mathcal{B}_\varepsilon(x)} \underbrace{\nabla\Gamma(x-y)\cdot\frac{x-y}{|x-y|}}_{=\frac{1}{n\omega_n}|x-y|^{-1}} f(y)\,\mathrm{d}y \\
&= \fint_{\partial\mathcal{B}_\varepsilon(x)} f(y)\,\mathrm{d}y \underset{\varepsilon\to 0^+}{\longrightarrow} f(x).
\end{aligned}$$

□

Remark 3. The condition $f \in \mathcal{C}_c^1(\mathbb{R}^n)$ is not optimal, however $f \in \mathcal{C}_c(\mathbb{R}^n)$ is not enough.

Proposition 1.8 (Green Formula)**.**

Let $U \subset \mathbb{R}^n$ be a open and bounded s.t. $\partial U \in \mathcal{C}^1$, $u \in \mathcal{C}^2(\overline{U})$ (i.e. $\partial_i u$ and $\partial_{ij} u$ are extendable on ∂U by continuity). For $x \in U$,

$$u(x) = \int_{\partial U} \left(u(y)\frac{\partial\Gamma}{\partial\eta}(y-x) - \Gamma(y-x)\frac{\partial u}{\partial\eta} \right) \mathrm{d}\sigma(y) - \int_U \Gamma(y-x)\Delta u(y)\,\mathrm{d}y,$$

where η is the exterior unit normal vector field w.r.t. U and $\frac{\partial}{\partial\eta} := \eta\cdot\nabla$.

Proof. Let $x \in U$, $\varepsilon > 0$, $\overline{\mathcal{B}_\varepsilon(x)} \subset U$ and $U_\varepsilon := U \setminus \mathcal{B}_\varepsilon(x)$. Using divergence formula, one gets [2]

$$\underbrace{\int_{U_\varepsilon} \Gamma(y-x)\Delta u(y)\,\mathrm{d}y}_{\underset{\varepsilon\to 0^+}{\longrightarrow} \int_U \Gamma(y-x)\Delta u(y)\,\mathrm{d}y} = \int_{\partial U_\varepsilon} \left(\Gamma(y-x)\frac{\partial u}{\partial\eta}(y) - \frac{\partial\Gamma}{\partial\eta}(y-x)u(y) \right) \mathrm{d}\sigma(y).$$

By theorem 1.7,

$$\int_{\partial\mathcal{B}_\varepsilon(x)} \Gamma(y-x)\frac{\partial u}{\partial\eta}(y)\,\mathrm{d}\sigma(y) \underset{\varepsilon\to 0^+}{\longrightarrow} 0.$$

For $y \in \partial\mathcal{B}_\varepsilon(x)$, $\eta(y) = -\frac{y-x}{|y-x|}$, hence

$$-\int_{\partial\mathcal{B}_\varepsilon(x)} \frac{\partial\Gamma}{\partial\eta}(y-x)u(y)\,\mathrm{d}\sigma(y) \underset{\varepsilon\to 0^+}{\longrightarrow} u(x).$$

Why don't we only consider $\partial\mathcal{B}_\varepsilon(x)$ and not ∂U in ∂U_ε ? □

[2] We use $\operatorname{div}(u\nabla v - v\nabla u)$ for well chosen functions.

Définition 1.9.

Let $U \subset \mathbb{R}^n$ be open and bounded. For $x \in U$, let $h^x \in \mathcal{C}^2(U) \cap \mathcal{C}^1(\overline{U})$ be the solution (if it exist) of

$$\begin{cases} \Delta h^x = 0 & \text{in } U \\ h^x = -\Gamma(\cdot - x) & \text{on } \partial U. \end{cases}$$

The Green function of U is defined by

$$\begin{aligned} G : U \times U \setminus \{(u,u) \mid u \in U\} &\longrightarrow \mathbb{R} \\ (x,y) &\longmapsto -\Gamma(y-x) - h^x(y). \end{aligned}$$

Remark 4. Existence is unclear but unicity comes from maximum principle.

Corolaire 1.10 (Green representation formula)**.**

Let $U \subset \mathbb{R}^n$ open and bounded, $\partial U \in \mathcal{C}^1$ and G the Green function of U. If $u \in \mathcal{C}^2(\overline{U})$ is a solution to

$$\begin{cases} -\Delta u = f & \text{in } U \\ u = g & \text{on } \partial U, \end{cases}$$

then for $x \in U$,

$$u(x) = \int_U G(x,y) f(y) \, \mathrm{d}y - \int_{\partial U} \underbrace{\frac{\partial G}{\partial \eta}(x,y)}_{=\eta(y)\cdot \nabla_y G(x,y)} \; g(y) \, \mathrm{d}\sigma(y).$$

Proof. Apply Green formula to h^x. □

Remark 5.

1. $K = -\frac{\partial G}{\partial \eta}$ is called *Poisson kernel for U*.
2. One can prove that $G \geq 0$ and $G(x,y) = G(y,x)$.

Proposition 1.11.

Let $n \geq 2$. Then, the Green function of $U = \mathcal{B}_1(0)$ is given by

$$G(x,y) = \begin{cases} \Gamma(|x|) \left(y - \frac{x}{|x|^2} \right) - \Gamma(y-x) & x \neq 0 \\ \Gamma(1) - \Gamma(y) & x = 0, \end{cases}$$

for all $(x,y) \in U \times U \setminus \{(u,u) \mid u \in U\}$, and Poisson kernel is given by

$$-\frac{\partial G}{\partial \eta}(x,y) = \frac{1 - |x|^2}{n\omega_n |x-y|^n}.$$

Proof.

Step 1 : We have that

$$h^x(y) = \begin{cases} -\Gamma\big(|x|(y - \bar{x}\big) & x \neq 0 \\ -\Gamma(1) & x = 0, \end{cases}$$

where $\bar{x} = \frac{x}{|x|^2}$. Indeed, we have that Γ is harmonic in $\mathbb{R}^n \setminus \{0\}$. Moreover if $x \in \mathcal{B}_1(0)$, then $\bar{x} \notin \mathcal{B}_1(0)$. For $y \in \partial\mathcal{B}_1(0)$, we have that

$$h^x(y) = -\Gamma(y - x),$$

since for $x \neq 0$,

$$\big||x|(y-\bar{x})^2\big| = |x|^2 \left(|y|^2 - \frac{2xy}{|x|^2} + \frac{1}{|x|^2}\right) = |x-y|^2.$$

Step 2 : For the Poisson kernel, we have that

$$\nabla\Gamma(y-x) = \frac{1}{n\omega_n} \frac{y-x}{|y-x|}.$$

Also,

$$\nabla_y h^x(y) = -\nabla_y \Gamma\Big(|x|(y-\bar{x})\Big)|x| = \frac{1}{n\omega_n}|x|^2 \frac{y-x}{|y-x|}.$$

Since $\eta(y) = y$,

$$\begin{aligned} -\frac{\partial G}{\partial \eta}(x,y) &= -\frac{\partial \Gamma}{\partial \eta}(y-x) - \frac{\partial h^x}{\partial \eta}(y) \\ &= -\frac{1}{n\omega_n}\left(\frac{(y-x)\cdot y}{|y-x|^n} - \frac{|x|^2 y\cdot y - x\cdot y}{|y-x|^2}\right). \end{aligned}$$

□

By translation and scaling, we get :

Théorème 1.12 (Poisson formula)**.**

Let $r > 0$, $x_0 \in \mathbb{R}^n$ and $g \in \mathcal{C}(\partial\mathcal{B}_r(x_0))$. Then,

$$u(x) = \begin{cases} \frac{r^2 - |x-x_0|^2}{n\omega_n r} \int_{\partial\mathcal{B}_r(x_0)} \frac{g(y)}{|x-y|^n}\,\mathrm{d}\sigma(y) & x \in \mathcal{B}_r(x_0) \\ g(x) & x \in \partial\mathcal{B}_r(x_0), \end{cases}$$

is a $\mathcal{C}^\infty(\mathcal{B}_r(x_0)) \cap \mathcal{C}\left(\overline{\mathcal{B}_r(x_0)}\right)$ function solving the Dirichlet problem

$$\begin{cases} \Delta u = 0 & \text{in } \mathcal{B}_r(x_0) \\ u = g & \text{on } \partial\mathcal{B}_r(x_0). \end{cases}$$

Proof. Prove that u is really a solution of the Dirichlet problem. Recall that Poisson kernel is given by

$$K = -\frac{\partial G}{\partial \eta}(x,y) = \frac{r^2 - |x-x_0|^2}{n\omega_n r}\frac{1}{|x-y|^n}.$$

If $y \in \partial\mathcal{B}_r(x_0)$ then $K(\cdot, y) \in \mathcal{C}^\infty(\mathcal{B}_r(x_0))$ and

$$\Delta u(x) = \int_{\partial\mathcal{B}_r(x_0)} \underbrace{\Delta_x K(x,y)}_{=0} g(y)\,\mathrm{d}\sigma(y) = 0,$$

because K is symmetric and harmonic w.r.t. y. Further by a discussion above

$$1 = \int_{\partial\mathcal{B}_r(x_0)} K(x,y)\,\mathrm{d}\sigma(y),$$

(otherwise, hand to compute, but it's not obvious). Lat $a \in \partial\mathcal{B}_r(x_0)$, and let $\varepsilon > 0$. Since g is continuous, there is $\delta > 0$ s.t. $|g(x) - g(a)| < \varepsilon$ whenever $|x - a| < \delta$. For $x \in \partial\mathcal{B}_r(x_0)$ and $|x - a| < \frac{\delta}{2}$,

$$|u(x) - g(a)| = \underbrace{\int_{\substack{\partial B_r(x_0)\\ |y-a|<\frac{\delta}{2}}} |K(x,y)||g(y) - g(a)| \,\mathrm{d}\sigma(y)}_{<\varepsilon} + \underbrace{\int_{\substack{\partial B_r(x_0)\\ |y-a|\geq\frac{\delta}{2}}} |K(x,y)||g(y) - g(a)| \,\mathrm{d}\sigma(y)}_{\leq \frac{2\|g\|_{L^\infty}}{n\omega_n r}\left(\frac{2}{\delta}\right)^n (r^2 - |x-x_0|^2) \underset{x\to\partial\mathcal{B}_r(x_0)}{\longrightarrow} 0} \underset{x\to\partial\mathcal{B}_r(x_0)}{\longrightarrow} \varepsilon.$$

□

Théorème 1.13.

Let $U \subset \mathbb{R}^n$ open. The following statements are equivalents

1. *$u \in \mathcal{C}^\infty(U)$ is harmonic,*
2. *$u \in \mathcal{C}(U)$ satisfy*

$$u(x) = \fint_{\mathcal{B}_r(x_0)} u(y) \,\mathrm{d}y,$$

for all $\mathcal{B}_r(x_0) \subset\subset U$.

Proof. **1.⇒2.** is the MVP. Let prove the converse. Let $\mathcal{B}_r(x_0) \subset\subset U$. By theorem 1.12, there exist $v \in \mathcal{C}^\infty(\mathcal{B}_r(x_0)) \cap \mathcal{B}(\overline{\mathcal{B}_r}(x_0))$ s.t.

$$\begin{cases} \Delta v = 0 & \text{in } \mathcal{B}_r(x_0) \\ v = u & \text{on } \partial\mathcal{B}_r(x_0). \end{cases}$$

The function $w = u - v$ satisfy the MVP and thus the maximum principal. Therefore, $w = 0$ and thus $u = v$ in $\mathcal{B}_r(x_0)$ and $u \in \mathcal{C}^\infty(\mathcal{B}_r(x_0))$. Since $\mathcal{B}_r(x_0)$ is unspecified, the claim follow. □

Remark 6. In the following, we say that $u \in \mathcal{C}(U)$ is subharmonic / superharmonic / harmonic if

$$u(x) \underset{\geq,=}{\leq} \fint_{\mathcal{B}_r(x_0)} u,$$

for all ball $\mathcal{B}_r(x_0) \subset\subset U$.

Théorème 1.14.

Let $U \subset \mathbb{R}^n$ be open and (u_n) a sequence of $\mathcal{C}(U)$ of harmonics functions. If (u_n) converge locally uniformly[a] *to $u : U \longrightarrow \mathbb{R}$, then $u \in \mathcal{C}^\infty(U)$ and u is harmonic.*

[a] (u_n) converge locally uniformly to u if (u_n) converge uniformly to u on all compacts sets.

Proof. Let $\mathcal{B}_r(x) \subset\subset U$. Then

$$u_n(x) \underset{n\to\infty}{\longrightarrow} u(x)$$

and

$$u_n(x) = \fint_{\mathcal{B}_r(x)} u \underset{n\to\infty}{\longrightarrow} \fint_{\mathcal{B}_r(x)} u.$$

Since u is continuous (by uniform convergence), the theorem follow from theorem 1.13. □

Théorème 1.15.

If $U \subset \mathbb{R}^n$ is open and $u \in \mathcal{C}(U)$ satisfy the MVP, then $u \in \mathcal{C}^\infty(U)$ and

$$|\partial^\alpha u(x)| \leq \frac{C_\alpha}{r^{n+k}} \|u\|_{L^1(\mathcal{B}_r(x))},$$

for all $\mathcal{B}r(x) \subset\subset U$ where

$$C_\alpha = \begin{cases} \frac{(2^{n+1}n|\alpha|)^{|\alpha|}}{\omega_n} & |\alpha| \geq 1 \\ \frac{1}{n\omega_n} & \alpha = 0. \end{cases}$$

Proof. If $\alpha = 0$, it's the MVP. Let $|\alpha| = 1$, says $\partial^\alpha = \partial_i$. Then $\Delta \partial_i u = 0$ and thus

$$\begin{aligned} |\partial_i u(x)| &\leq \left| \fint_{\mathcal{B}_{r/2}} \partial_i u \right| \\ &\underset{\text{IBP}}{=} \frac{2^n}{\omega_n r^n} \left| \int_{\partial \mathcal{B}_{r/2}(x)} u \eta_i \, \mathrm{d}\sigma \right| \\ &\leq \frac{2n}{r} \sup_{y \in \partial \mathcal{B}_{\frac{r}{2}}(x)} |u(y)|. \end{aligned}$$

Since $\mathcal{B}_{\frac{r}{2}}(y) \subset \mathcal{B}_r(x)$, we finally get

$$|u(y)| \leq \frac{1}{n\omega_n} \left(\frac{2}{r}\right)^n \|u\|_{L^1(\mathcal{B}_r(x))},$$

and thus, the claim follow. □

Corolaire 1.16.

Every harmonic functions is real analytic.

Proof. Taylor's formula. □

Corolaire 1.17 (Liouville)**.**

Let $u : \mathbb{R}^n \longrightarrow \mathbb{R}$ be harmonic and bounded. Then u is constant.

Proof. Let $x \in \mathbb{R}^n$ and $r > 0$. By theorem 1.15,

$$\|\nabla u(x)\| \leq \frac{2^{n+1}n}{r^{n+1}} \|u\|_{L^1(\mathcal{B}_r(x))} \leq \frac{2^{n+1}n}{r} \sup_{\mathbb{R}^n} |u| \underset{n \to \infty}{\longrightarrow} 0.$$

□

Théorème 1.18 (Harnack's convergence theorem)**.**

Let $U \subset \mathbb{R}^n$ be a domain. If (u_n) is an increasing sequence of harmonic functions which converge at some point, then (u_n) converge locally uniformly to a harmonic function.

Proof. Let $V \subset\subset U$ be a subdomain. Suppose WLOG that $(u_n(y))_n$ converge for some $y \in V$. Let $\varepsilon > 0$. By Harnack's inequality $(n \geq m)$,

$$\sup_V |u_n - u_m| \leq C|u_n(y) - u_m(y)| < \varepsilon,$$

if $n \geq m \geq n_0$ for a certain $n_0 \in \mathbb{N}$. Therefore, (u_n) is a Cauchy sequence in $\mathcal{C}(\overline{V})$. The claim follow by theorem 1.14. □

Définition 1.19.

Let $U \subset \mathbb{R}^n$ be open. We define

$$\sigma(U) = \{u : U \longrightarrow \mathbb{R} \mid u \text{ is subharmonic}\},$$

and for $g : \partial U \longrightarrow \mathbb{R}$ given,

$$\sigma(U,g) = \{u : U \longrightarrow \mathbb{R} \mid u \leq g \text{ on } \partial U\}.$$

Remark 7. If $u \in \sigma(U)$ and if $x \in U$ is s.t. $u(x) = \sup_{y \in U} u(y)$, then u is constant.

Lemme 1.20.

Let $U \subset \mathbb{R}^n$ be open. Then

1. *If $u_1, \ldots, u_N \in \sigma(U)$ and $\lambda_j \geq 0$, then $\sum_{i=1}^{N} \lambda_i u_i \in \sigma(U)$,*
2. *If $u_1, u_2 \in \sigma(U)$, then $\max\{u_1, u_2\} \in \sigma(U)$,*
3. *If $f : \mathbb{R} \longrightarrow \mathbb{R}$ is convex and increasing and $u \in \sigma(u)$, then $f(u) \in \sigma(U)$.*

Proof. **1.** is clear by linearty of the integral. **3.** is clear by Jensen's inequality. For **2.**, let $x \in U$ and $\mathcal{B}_r(x) \subset U$. Suppose $u_1(x) \geq u_2(x)$. Then,

$$\max\{u_1(x), u_2(x)\} = u_1(x) \leq \fint_{\partial B_r(x)} u_1 \, \mathrm{d}\sigma \leq \fint_{\partial \mathcal{B}_r(x)} \max\{u_1, u_2\} \, \mathrm{d}\sigma.$$

□

Définition 1.21.

Let $U \subset \mathbb{R}^n$ be open, $u \in \mathcal{C}(U)$, $x_0 \in U$ and $r > 0$ s.t. $\mathcal{B}_r(x_0) \subset\subset U$. Then, the harmonic lift of u is defined by

$$u_{x_0,r}(x) = \begin{cases} \frac{r^2 - |x-x_0|^2}{n\omega_n r} \int_{\partial \mathcal{B}_r(x_0)} \frac{u(x)}{|x-y|^n} \, \mathrm{d}\sigma(y) & x \in \mathcal{B}_r(x_0) \\ u(x) & \textit{otherwise.} \end{cases}$$

Remark 8.

1. $u_{x_0,r} \in \mathcal{C}^\infty(\mathcal{B}_r(x_0)) \cap \mathcal{C}(\overline{\mathcal{B}_r(x_0)})$ is harmonic in $\mathcal{B}_r(x_0)$ and coincide with u on $\partial \mathcal{B}_r(x_0)$.
2. If $u, v \in \mathcal{C}(U)$, are s.t. $v \leq u$ and $\mathcal{B}_r(x_0) \subset\subset U$, then $v_{x_0,r} \leq u_{x_0,r}$.

Lemme 1.22.

Let $U \subset \mathbb{R}^n$ open, $u \in \sigma(U)$ and $\mathcal{B}_r(x_0) \subset\subset U$. Then,

1. *$u \leq u_{x_0,r}$,*
2. *$u_{x_0,r} \in \sigma(U)$.*

Proof. **1.** Since $u_{x_0,r}$ is harmonic, on $\mathcal{B}_r(x_0)$, the function $u - u_{x_0,r} \in \sigma(U)$. Apply the maximum principle to $u - u_{x_0,r}$ on $\mathcal{B}_r(x_0)$, and thus $u - u_{x_0,r} \leq 0$ in $\mathcal{B}_r(x_0)$. The claim follow since $u_{x_0,r} = u$ on $U \setminus \mathcal{B}_r(x_0)$.

2. We want to show that if $\mathcal{B}_\rho(\xi) \subset\subset U$, then

$$u_{x_0,r}(\xi) \leq \int_{\partial \mathcal{B}_\rho(\xi)} u_{x_0,r} \, \mathrm{d}\sigma.$$

Step 1 : Suppose $\xi \notin \mathcal{B}_r(x_0)$. Then,

$$u_{x_0,r}(\xi) = u(\xi) \leq \fint_{\partial \mathcal{B}_\rho(\xi)} u \, \mathrm{d}\sigma \underset{\mathbf{1.}}{\leq} \fint_{\partial \mathcal{B}_\rho(\xi)} u_{x_0,r} \, \mathrm{d}\sigma.$$

Step 2 : Suppose $\xi \in \mathcal{B}_r(x_0)$. Assume that $u_{x_0,r} \notin \sigma(U)$. Then, there is ρ and ξ s.t.

$$u_{x_0,r}(\xi) > \fint_{\partial \mathcal{B}_\rho(\xi)} u_{x_0,r} \, \mathrm{d}\sigma. \tag{1.1}$$

Set $v := (u_{x_0,r})_{\xi,\rho}$. Then, v is harmonic in $\mathcal{B}_\rho(\xi)$. So,

$$v(\xi) = \fint_{\partial \mathcal{B}_\rho(\xi)} u_{x_0,r} \, \mathrm{d}\sigma \underset{(1.1)}{<} u_{x_0,r}. \tag{1.2}$$

Since $u_{x_0,r} - v$ is harmonic in the domain $D = \mathcal{B}_\rho(\xi) \cap \mathcal{B}_r(x_0)$, by the maximum principle and (1.2), there is $p \in \partial D$ s.t. $u_{x_0,r} - v(p) > 0$. If $p \in \partial \mathcal{B}_\rho(\xi) \cap \overline{\mathcal{B}_r(x_0)}$, then $u_{x_0,r}(p) = v(p)$ which is a contradiction. If $p \in \partial \mathcal{B}_r(x_0) \cap \overline{\mathcal{B}_\rho(\xi)}$, then $v(p) < u_{x_0,r}(p) = u(p)$. By the remark and **1.**, $v \geq u_{\xi,\rho} \geq u$ in U which is also a contradiction. The claim follow.

□