EXERCISES OPERATOR THEORY

SHEET 9

Exercise 33

Show: $A, B \in M_n(\mathbb{C})$ commute $\Rightarrow e^{A+B} = e^A e^B$.

Reminder.

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!}$$

To show the claim, we need the binomial theorem for matrices:

Lemma. *If $A, B \in M_n(\mathbb{C})$ **commute**, then*

$$(A+B)^k = \sum_{l=0}^{k} \binom{k}{l} A^l B^{k-l}$$

Proof. Induction, see e.g. Forster, Analysis 1. □

Proof of exercise: Calculation:

$$\begin{aligned} e^A e^B &= \sum_{l=0}^{\infty} \frac{A^l}{l!} \sum_{m=0}^{\infty} \frac{B^m}{m!} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{A^l B^m}{l! m!} = \sum_{l=0}^{\infty} \sum_{m=0}^{\infty} \frac{(l+m)!}{l! m!} \frac{A^l B^m}{(l+m)!} \\ &= \sum_{l=0} \sum_{k=l}^{\infty} \frac{k!}{l!(k-l)!} \frac{1}{k!} A^l B^{k-l} = \sum_{k=0}^{\infty} \sum_{l=0}^{k} \binom{k}{l} \frac{1}{k!} A^l B^{k-l} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{n=0}^{k} \binom{k}{n} A^l B^{k-l} \\ &= \sum_{k=0}^{\infty} \frac{1}{k!} (A+B)^k = e^{A+B} \end{aligned}$$

□

Exercise 34

Find the semigroup generated by

(a)

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$

(b)

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Reminder. *(Definition 1.6, second part) The semigroup generated by $A \in M_n(\mathbb{C})$ is given by e^{tA}.*

In both parts, let Id denote the corresponding unit matrix.

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(a) We have

$$A^2 = \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

And hence $A^n = 0$ for all $n \geq 2$. This means

$$e^{tA} = \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \mathrm{Id} + tA$$

Or as matrix,

$$e^{tA} = \begin{pmatrix} 1+t & t \\ -t & 1-t \end{pmatrix}$$

(b) The matrix A can be written as

$$\begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: D + N$$

We see easily that $N^2 = 0$, therefore $e^{tN} = \mathrm{Id} + tN$. Furthermore, D and N commute and

$$e^{tD} = \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix}$$

Hence, by exercise 33,

$$\begin{aligned} e^{tA} = e^{tD} e^{tN} &= \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} (\mathrm{Id} + tN) \\ &= \begin{pmatrix} e^{2t} & 0 & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} + \begin{pmatrix} 0 & te^{2t} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} e^{2t} & te^{2t} & 0 \\ 0 & e^{2t} & 0 \\ 0 & 0 & e^t \end{pmatrix} \end{aligned}$$

Remark. *The matrices in this exercise were "nice" in a sense that no additional calculations were needed. In general, on needs to find the Jordan (or diagonal) form of the matrix, $A = BJB^{-1}$ and make use of the identity*

$$e^{tA} = Be^{tJ}B^{-1}.$$

Exercise 35

Prove for $e^{tA}, A \in M_n(\mathbb{C})$ that the following are equivalent:

(a) The semigroup is bounded, i.e. there exists $M \geq 1$ such that $\|e^{tA}\| < M$ for all $t \geq 0$.

(b) All eigenvalues λ of A satisfy $\Re\lambda \leq 0$ and whenever $\Re\lambda = 0$, then λ is a simple eigenvalue (i.e. the Jordan blocks coresponding to λ have size 1)

Remark. *Since all norms on a finite-dimensional space are equivalent, we can choose a norm convenient to us.*

Proof. "$\Rightarrow$": Assume that e^{tA} is bounded, but there exists an eigenvalue λ_0 with $\Re\lambda_0 > 0$. Then for $A = BJB^{-1}$, we have

$$e^{tA} = Be^{tJ}B^{-1} \Leftrightarrow B^{-1}e^{tA}B = e^{tJ}.$$

Therefore, for $\|A\| := \max_{1\leq i,j\leq n} |a_{ij}|$,

$$\begin{aligned}\|B^{-1}\|\|e^{tA}\|\|B\| &\geq \|B^{-1}e^{tA}B^{-1}\| = \|e^{tJ}\| = \|e^{tD}e^{tN}\| \\ &= \|e^{tD}[\operatorname{Id} + (\text{off-diag. terms})]\| \geq |e^{t\lambda_0}| \to \infty, t \to \infty\end{aligned}$$

and hence $\|e^{tA}\| \to \infty, t \to \infty$, which contradicts our assumption that e^{tA} is bounded.

Assume that there exists λ_1 with $\Re\lambda_1 = 0$ such that the Jordan form has the following representation:

$$J = \left(\begin{array}{cc|c} \lambda_1 & 1 & \multirow{2}{*}{0} \\ 0 & \lambda_1 & \\ \hline \multicolumn{2}{c|}{0} & C \end{array}\right)$$

where C is the rest of the Jordan matrix. Denote the upper block by J_1. Then with the same norm as above,

$$\begin{aligned}\|B^{-1}\|\|e^{tA}\|\|B\| &\geq \|e^{tJ}\| \geq \|e^{tJ_1}\| = \|e^{tD}e^{tN}\| \\ &= \left\| \begin{pmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_1 t} \end{pmatrix} \left[\operatorname{Id} + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \right\| = \left\| \begin{pmatrix} e^{\lambda_1 t} & te^{\lambda_1 t} \\ 0 & e^{\lambda_1 t} \end{pmatrix} \right\| \\ &\geq |te^{t\lambda_1}| \to \infty, t \to \infty\end{aligned}$$

since $|e^{t\lambda_1}| = 1$. The calculation for larger blocks works similarly.

"$\Leftarrow$": Assume that $\Re\lambda \leq 0$ for all eigenvalues of A and if $\Re\lambda = 0$, λ is simple. Let l be the number of Jordan blocks, denoted by J_i. Then since the Jordan blocks (seen as sub-matrices) commute:

$$\|e^{tA}\| = \|Be^{tJ}B^{-1}\| \leq \underbrace{\|B\|\|B^{-1}\|}_{=:\beta} \|e^{tJ}\| \leq \beta \prod_{i=1}^{l} \|e^{tJ_i}\|$$

Hence, it suffices to show that each Jordan block is bounded. Let us distinguish the cases for different-sized Jordan blocks:

(a) Block of size one: We have $\Re\lambda_i \leq 0$, hence

$$\|e^{tJ_i}\| = |e^{t\lambda_i}| \leq 1$$

(b) Block of size larger than one: We already know that $\Re\lambda_i < 0$. Consider the case where the size is equal to 2:

$$\begin{aligned}\|e^{tJ_i}\| &= \|e^{tD_i}e^{tN_i}\| \\ &= \left[\begin{pmatrix} e^{\lambda_i t} & 0 \\ 0 & e^{\lambda_i t} \end{pmatrix} \left[\operatorname{Id} + t \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\right] \right\| = \left\| \begin{pmatrix} e^{\lambda_i t} & te^{\lambda_i t} \\ 0 & e^{\lambda_i t} \end{pmatrix} \right\| \\ &\leq \max\{e^{t\lambda_i}, te^{t\lambda_i}\} \to 0, t \to \infty\end{aligned}$$

For Jordan blocks of larger size, the calculations are similar, but the polynomials appearing will be of higher order. □

Exercise 36

Let M_q with maximal domain $D(M_q)$ be the multiplication operator on $C_0(\Omega)$ induced by some continuous function q. Show:

(a) M_q has a bounded inverse $\Leftrightarrow$ q has a bounded inverse $\frac{1}{q}$, i.e. $0 \notin \overline{q(\Omega)}$. In this case, we have

$$M_q^{-1} = M_{\frac{1}{q}} \tag{1}$$

(b) $\sigma(M_q) = \overline{q(\Omega)}$.

Proof. (a) Recall that the norm $\|\cdot\|$ on $C_0(\Omega)$ is the supremum norm. Also, let us recall us a version of Uryson's Lemma:

Lemma. *Let $A, B \subset \Omega$ closed and disjoint. Then there exists a continuous function $f\colon \Omega \to [0,1]$ such that $f \equiv 0$ on A and $f \equiv 1$ on B.*

"$\Leftarrow$": Let $0 \notin \overline{q(\Omega)}$. Since q is continuous, it is bounded away from zero and hence, $\frac{1}{q}$ is bounded and continuous. In this case, we obviously have (1) and

$$\left\| M_{\frac{1}{q}} \right\| = \sup_{\|f\|=1} \left\| \frac{f}{q} \right\| \leq \left\| \frac{1}{q} \right\| < \infty$$

since $\frac{1}{q}$ is bounded.

"$\Rightarrow$" Assume that M_q has a bounded inverse M_q^{-1}. Then for $f \in D(M_q)$, we have

$$\|f\| = \|M_q^{-1} M_q f\| \leq \|M_q^{-1}\| \|M_q f\|.$$

Consider $f \in D(M_q)$ with $\|f\| = 1$, then

$$\delta := \frac{1}{\|M_q^{-1}\|} \leq \|M_q f\| = \sup_{s \in \Omega} |q(s) f(s)| \tag{2}$$

We want to show that q is bounded away from zero. In particular, we want to show $|q| \geq \frac{\delta}{2}$.

Assume that $\inf_{s \in \Omega} |q(s)| < \frac{\delta}{2}$. Since q is continuous, there exists an open set $\mathcal{O} \subset \Omega$ such that $q(s) < \frac{\delta}{2}$ for all $s \in \mathcal{O}$. On the other hand, by Uryson's Lemma, we find a function $f_0 \in C_0(\Omega)$ such that $f_0 \equiv 0$ on $\Omega \setminus \mathcal{O}$ and $f_0 \equiv 1$ on some $K \subset \mathcal{O}$ compact $\Rightarrow \|f_0\| = 1$. Therefore,

$$\delta \overset{(2)}{\leq} \sup_{s \in \Omega} |q(s) f_0(s)| = \sup_{s \in \mathcal{O}} |\underbrace{q(s)}_{\leq \frac{\delta}{2}} \underbrace{f_0(s)}_{\leq 1}| \leq \frac{\delta}{2}$$

which is a contradiction. Hence, $\inf_{s \in \Omega} |q(s)| \geq \frac{\delta}{2}$ and therefore $0 \notin \overline{q(\Omega)}$. The representation (1) is obvious again.

(b) We have $\lambda - M_q = M_{\lambda - q}$. Hence, the claim follows if we apply a) to the operator $M_{\lambda - q}$:

$$\lambda \in \sigma(M_q) \Leftrightarrow 0 \in \sigma(M_{\lambda - q}) \Leftrightarrow M_{\lambda - q} \text{ is not boundedly invertible}$$
$$\overset{a)}{\Leftrightarrow} 0 \in \overline{(\lambda - q)(\Omega)} \Leftrightarrow \lambda \in \overline{q(\Omega)}$$

$\square$