EXERCISES OPERATOR THEORY

SHEET 7

Exercise 25

Let H be a Hilbert space. Show that for a bounded operator, the numerical range is convex.

Reminder. *For an operator $B \in \mathcal{B}(H)$ on a Hilbert space H, the numerical range is defined as*

$$\operatorname{Num}(B) := \{(Bx, x)\colon \|x\| = 1\}$$

Proof. The proof (in a more general version) can be found in the source given in the lecture notes. Hence, there will only be a short sketch given here. Assuming $\operatorname{Num}(B)$ consists of more than one point, we can find two distinct elements $\alpha_1, \alpha_2 \in \operatorname{Num}(B)$ and $x_1, x_2 \in H$ such that

$$\alpha_1 = (Bx_1, x_1) \text{ and } \alpha_2 = (Bx_2, x_2).$$

We need to show that the line connecting α_1 and α_2 is also in $\operatorname{Num}(B)$. The idea is to construct a function g_1 which continuously maps some "path" from x_1 to x_2 to the line $[\alpha_1, \alpha_2] \subset \mathbb{C}$. As a first step, we may reduce the problem to $[\alpha_1, \alpha_2] = [0, 1]$ by modifying the function g_1 to g. Next, we need to find the right parameters of the function such that g actually maps to the right interval, especially $g(0) = 0$ and $g(1) = 1$, hence, by the intermediate value theorem, the claim follows. □

Exercise 26

Let X be a Banach space with a Schauder basis. Show that X is separable.

Reminder.

- *A subset $\{e_n\}_{n\in\mathbb{N}}$ of a Banach space X is called a Schauder basis is every $x \in X$ has a unique representation of the form*

$$x = \sum_{n=1}^{\infty} \lambda_n e_n.$$

- *Note that the converse is not true (cf. remark in lecture notes).*

Proof. The proof works similarly to the proof of exercise 21 on sheet 6. □

Exercise 27

Let H_1, H_2 be Hilbert spaces. Show that $HS(H_1, H_2)$ is a subspace of $\mathcal{B}(H_1, H_2)$ and $\|\cdot\|_{HS}$ is a norm on $HS(H_1, H_2)$.

Reminder. *Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system of H_1. The Hilbert-Schmidt norm of an operator $T \in \mathcal{B}(H_1, H_2)$ is defined as*

$$\|T\|_{HS} := \left(\sum_{n=1}^{\infty} \|Te_n\|_{H_1}^2\right)^{\frac{1}{2}} = \|\{\|Te_n\|_{H_1}\}_{n\in\mathbb{N}}\|_{l^2}.$$

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Note that the norm is independent of the choice of the orthonormal system and hence well-defined. Also note that by formula (4.12),

$$\|T\| \le \|T\|_{HS}.$$

The space of all such operators with finite Hilbert-Schmidt norm is denoted by $HS(H_1, H_2)$.

Proof. Let us show first that $HS(H_1, H_2)$ is a linear space. The inclusion $HS(H_1, H_2) \subset \mathcal{B}(H_1, H_2)$ is already given by definition, or alternatively, follows from formula (4.12).

It is obvious from the definition that the zero operator $0 \in \mathcal{B}(H_1, H_2)$ has zero norm. Hence, $0 \in HS(H_1, H_2)$.

Let $\lambda \in \mathbb{F}, T_1, T_2 \in HS(H_1, H_2)$. Then by the triangle inequality of the operator norm as well as the l^2-norm,

$$\begin{aligned}\|\lambda T_1 + T_2\|_{HS} &= \|\{\|(\lambda T_1 + T_2)e_n\|\}_{n\in\mathbb{N}}\|_{l^2} \\ &\le |\lambda|\, \|\{\|T_1 e_n\|\}_{n\in\mathbb{N}}\|_{l^2} + \|\{\|T_2 e_n\|\}_{n\in\mathbb{N}}\|_{l^2} \\ &= |\lambda| \|T_1\|_{HS} + \|T_2\|_{HS} < \infty\end{aligned}$$

Therefore, $\lambda T_1 + T_2 \in HS(H_1, H_2)$.

For the proof of the norm properties, the above calculation already covers the triangle inequality by setting $\lambda = 1$. For homogeneity, we can repeat the calculation with $T_2 = 0$. The inequality turns into an equality and we're done.

For definiteness, one uses formula (4.12). Assume that we have $\|T\|_{HS} = 0$ for some operator $T \in HS(H_1, H_2)$. Then by (4.12), also $\|T\| = 0$ and hence, $T = 0$. Alternatively, one can use $\|T\|_{HS} = 0$ directly to obtain $\|Te_n\|_{H_1} = 0$ for all $n \in \mathbb{N}$, which also implies $T = 0$. $\square$

Exercise 28

Let H be a Hilbert space. Show:

(1) $B \in HS(H) \Leftrightarrow B^* \in HS(H)$.
(2) $A \in HS(H), B \in \mathcal{B}(H) \Rightarrow AB, BA \in HS(H)$.

Reminder (Parseval's identity)**.** *Let $\{e_n\}_{n\in\mathbb{N}}$ be a complete orthonormal system in H.*

$$\|x\|^2 = \sum_{n_1}^{\infty} |(x, e_n)|^2 \ \forall x \in H. \tag{1}$$

Proof. For both parts, let $\{e_n\}_{n\in\mathbb{N}}$ complete ONS for H.

(1) We show that the Hilbert-Schmidt norms of B and B^* coincide, which shows both directions at once.

$$\begin{aligned}\|B\|_{HS}^2 &= \sum_{n=1}^{\infty} \|Be_n\|^2 \overset{(1)}{=} \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} |(Be_n, e_m)|^2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |(Be_n, e_m)|^2 \\ &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} |(e_n, B^* e_m)|^2 \overset{(1)}{=} \sum_{m=1}^{\infty} \|B^* e_m\|^2 = \|B^*\|_{HS}^2\end{aligned}$$

(2) Show first that $BA \in HS(H)$ if $A \in HS(H)$ $(*)$.

$$\|BA\|_{HS}^2 = \sum_{n=1}^{\infty} \|BAe_n\|^2 \le \sum_{n=1}^{\infty} \|B\|^2 \|Ae_n\|^2 = \|B\|^2 \sum_{n=1}^{\infty} \|Ae_n\|^2 = \|B\|^2 \|A\|_{HS}^2 < \infty$$

Hence, the Hilbert-Schmidt property is preserved by left multiplication with a general bounded operator. ($HS(H)$ is a left ideal). Note that for the other direction, the above estimate does not work directly. But, on the other hand,

$$A \in HS(H) \overset{(1)}{\Rightarrow} A^* \in HS(H) \overset{(*)}{\Rightarrow} B^*A^* \in HS(H) \overset{(1)}{\Rightarrow} AB = (B^*A^*)^* \in HS(H)$$

$\square$