EXERCISES OPERATOR THEORY

SHEET 2

Exercise 5

$X = C[0,1]$, operator A on X given by

$$Ax(t) = x(0) + tx(1).$$

Find $\sigma(A), r_\sigma(A)$ and $R_\lambda(A)$ for $\lambda \in \rho(A)$.

Spectrum $\sigma(A)$:

(1) $0 \in \sigma(A)$: Consider the function $x(t) = t(1-t)$. Then $Ax(t) = 0$. Hence, $\mathrm{Ker}(A) \neq \{0\}$ and A is not injective. Therefore, $0 \in \sigma(A)$.

(2) $1 \in \sigma(A)$: Consider the function $x(t) = t$. Then $(A - \mathrm{Id})x(t) = 0$. By the same argument as before, $1 \in \sigma(A)$.

Resolvent set $\rho(A)$ and Resolvent $R_\lambda(A)$: $\lambda \in \rho(A)$ for all $\lambda \notin \{0,1\}$: We can calculate the inverse directly: Let $y \in C[0,1]$. We need to find $x \in C[0,1]$ such that

$$(A-\lambda)x(t) = x(0) + tx(t) - \lambda x(t) = y(t). \tag{1}$$

For $t = 0$, we need

$$y(0) = x(0) - \lambda x(0) = (1-\lambda)x(0) \Leftrightarrow x(0) = \frac{1}{1-\lambda}y(0)$$

For $t = 1$, we need

$$y(1) = x(0) + x(1) - \lambda x(1) = x(0) + (1-\lambda)x(1) = \frac{1}{1-\lambda}y(0) + (1-\lambda)x(1)$$

$$\Leftrightarrow x(1) = \frac{1}{1-\lambda}y(1) - \frac{1}{(1-\lambda)^2}y(0)$$

Hence, we can rewrite equation (1) as

$$\frac{1}{1-\lambda}y(0) + \frac{t}{1-\lambda}y(1) - \frac{t}{(1-\lambda)^2}y(0) - \lambda x(t) = y(t)$$

$$\Leftrightarrow x(t) = -\frac{1}{\lambda}y(t) + \frac{1}{\lambda(1-\lambda)}\left[y(0) + ty(1) - \frac{t}{1-\lambda}y(0)\right]$$

Therefore, the inverse is given by

$$(A-\lambda)^{-1}y(t) = -\frac{1}{\lambda}y(t) + \frac{1}{\lambda(1-\lambda)}\left[y(0) + ty(1) - \frac{t}{1-\lambda}y(0)\right]$$

which is also the explicit form of the resolvent. (One can now double-check this by plugging in $y = (A-\lambda)x$ to see that this is in fact the inverse.)

$\Rightarrow A - \lambda$ is invertible $\Rightarrow \lambda \in \rho(A)$. As stated before, we also have $R_\lambda(A) = (A-\lambda)^{-1}$ with the above representation.

Spectral radius $r_\sigma(A)$: We obtain the spectral radius directly by definition,

$$r_\sigma(A) = 1.$$

Corrections, comments to peter.kuchling@uni-bielefeld.de.

How does one actually find the spectrum? One idea is to just write out the equation to get an idea how the spectrum could look like: For some $\lambda \in \mathbb{C}$, we need

$$x(0) + tx(1) = \lambda x(t). \tag{2}$$

We know that for $t = 0$, (2) becomes

$$x(0) = \lambda x(0) \Leftrightarrow \lambda = 1 \vee x(0) = 0$$

Consider the case $\lambda = 1$:

$$x(0) + tx(1) = x(t)$$

For $t = 1$, we obtain $x(0) + x(1) = x(1)$ and see that $x(0) = 0$. We are left with

$$x(t) = tx(1)$$

so the eigenfunction is some linear function.

Now to the case that $\lambda \neq 1$. By above considerations at $t = 0$, we know that $x(0) = 0$ is necessary now. This leaves us with the equation

$$tx(1) = \lambda x(t).$$

at $t = 1$, this becomes $x(1) = \lambda x(1)$. Since we excluded the case $\lambda = 1$, we conclude that $\lambda = 0$ or $x(1) = 0$. As it turns out, both conditions are equivalent in our setting. Hence, we only have to find some continuous function for which $x(0) = x(1) = 0$ holds (e.g. $x(t) = t - t^2$) and we have an eigenfunction to the eigenvalue $\lambda = 0$.

Of course, in a more general setting, not all elements of the spectrum are eigenvalues, and this becomes more complicated. But as a first step, one can try to explicitly calculate the spectrum.

Exercise 6

Let X be a Banach space. Suppse $A \in \mathcal{B}(X)$ and $\lambda \in \sigma(A)$. Show that $\lambda^n \in \sigma(A^n)$ for all $n \in \mathbb{N}$.

Reminder (Spectral Mapping Theorem)**.** *Let f be analytic in some neighbourhood of $\sigma(B)$. Then*

$$\sigma(f(B)) = f(\sigma(B))$$

Obviously, the function $f(z) = z^n$ is analytic for all $n \in \mathbb{N}$. Hence, the statement of the exercise follows by the spectral mapping theorem.

Alternatively, we can use an argument from the proof of Theorem 2.23: Assume $\lambda \in \sigma(A)$. Then $A - \lambda$ is not invertible. We can rewrite $(A^n - \lambda^n)$ as

$$A^n - \lambda^n = (A - \lambda)(\lambda^{n-1} + \lambda^{n-2}A + \cdots + \lambda A^{n-2} + A^{n-1})$$

Since the operators on the right-hand-side commute, by Lemma 2.3, the operator $A^n - \lambda^n$ is not invertible. Hence, $\lambda^n \in \sigma(A^n)$.

Exercise 7

$A\colon l^2 \to l^2, Ax = (\lambda_1 x_1, \lambda_2 x_2, \dots), \lambda_n \in \mathbb{C}, \sup|\lambda_n| < \infty$. Find $\sigma(A)$.

Claim: $\sigma(A) = \overline{\{\lambda_n\}}_{n\in\mathbb{N}}$.

(1) $\lambda_n \in \sigma(A)\ \forall n \in \mathbb{N}$.
Let $e_n = (0, \dots, 0, 1, 0, \dots)$. Then

$$Ae_n = (0, \dots, 0, \lambda_n, 0 \dots) = \lambda_n e_n.$$

Hence, λ_n is an eigenvalue of A and by Lemma 2.7, $\lambda_n \in \sigma(A)$.

(2) Let $\lambda \in \overline{\{\lambda_n\}}_{n\in\mathbb{N}} \setminus \{\lambda_n\}_{n\in\mathbb{N}}$. Then there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ such that $\lambda_{n_k} \to \lambda, k \to \infty$. Especially,

$$\sup_{k\in\mathbb{N}} \left| \frac{1}{\lambda_{n_k} - \lambda} \right| = \infty.$$

Assume that $A-\lambda$ is invertible. Then the inverse is given by

$$(A-\lambda)^{-1}y = \left(\frac{1}{\lambda_1-\lambda}y_1, \frac{1}{\lambda_2-\lambda}y_2, \dots\right)$$

we show that this operator would be unbounded:

$$\|(A-\lambda)^{-1}\| = \sup_{\|y\|=1}\sum_{n=1}^{\infty}\left(\frac{y_n}{\lambda_n-\lambda}\right)^2 \geq \sup_{j\in\mathbb{N}}\sum_{n=1}^{\infty}\frac{(e_j)_n^2}{(\lambda_n-\lambda)^2} = \sup_{j\in\mathbb{N}}\frac{1}{(\lambda_j-\lambda)^2} = \infty$$

By theorem 2.2 (Banach), a bounded invertible operator has bounded inverse. Hence, $A-\lambda$ can not be invertible.

(3) $\lambda \notin \sigma(A)$ for all $\lambda \notin \overline{\{\lambda_n\}}_{n\in\mathbb{N}}$.
Let $\lambda \notin \{\lambda_n\}_{n\in\mathbb{N}}$, i.e. $\lambda \neq \lambda_n$ $\forall n \in \mathbb{N}$. We claim that $A-\lambda$ is invertible. By theorem 2.2, it suffices to show that the operator $A-\lambda$ is bijective.

- $A-\lambda$ is injective: For any $x \in l^2$, we have

$$(A-\lambda)x = ((\lambda_1-\lambda)x_1, (\lambda_2-\lambda)x_2, \dots)$$

Then

$$(A-\lambda)x = 0 \Leftrightarrow (\lambda_n-\lambda)x_n = 0 \ \forall n\in\mathbb{N} \overset{\lambda_n\neq\lambda}{\Leftrightarrow} x_n = 0 \ \forall n \in \mathbb{N}$$

Hence, $\mathrm{Ker}(A-\lambda) = \{0\}$ and $A-\lambda$ is injective.

- $A-\lambda$ is surjective: Let $y \in l^2$. We need to find $x \in l^2$ such that $(A-\lambda)x = y$. This means that

$$(\lambda_n-\lambda)x_n = y_n \ \forall n\in\mathbb{N} \overset{\lambda_n\neq\lambda}{\Leftrightarrow} x_n = \frac{1}{\lambda_n-\lambda}y_n$$

Hence, we found the desired x and $A-\lambda$ is surjective. It is still left to show that $x \in l^2$. But since $\lambda \notin \overline{\{\lambda_n\}}_{n\in\mathbb{N}}$, there exists $\varepsilon > 0$ such that

$$(\lambda-\varepsilon, \lambda+\varepsilon) \cap \overline{\{\lambda_n\}}_{n\in\mathbb{N}} = \emptyset \Rightarrow \sup_{n\in\mathbb{N}}|\lambda_n-\lambda| > \varepsilon \Rightarrow \sup_{n\in\mathbb{N}}\left|\frac{1}{\lambda_n-\lambda}\right| < \frac{1}{\varepsilon}$$

And hence,

$$\|x\|_{l^2} \leq \frac{1}{\varepsilon}\|y\|_{l^2} \Rightarrow x \in l^2$$

and $A-\lambda$ is surjective.

In total, $A-\lambda$ is bijective and hence by Theorem 2.2 invertible.

Remark. *The inverse is given by*

$$(A-\lambda)^{-1}y = \left(\frac{1}{\lambda_1-\lambda}y_1, \frac{1}{\lambda_2-\lambda}y_2, \dots\right)$$

Exercise 8

Consider the operator

$$Ax(t) = \int_a^t x(u)du$$

on $(C[a,b], \|\cdot\|_\infty)$. Show that $\sigma(A) = 0$.

(1) $\lambda = 0 \in \sigma(A)$.
Side note: By the fundamental theorem of calculus, A is injective. But A is not surjective: Since by definition, $Ax(0) = 0$ for all $x \in C[a,b]$, there exists no x such that $Ax = y$ for e.g. $y \equiv 1$. Hence, A is not invertible and $0 \in \sigma(A)$.

(2) $\lambda \notin \sigma(A)$ for all $\lambda \neq 0$.

Let $\lambda \neq 0$ and consider the equation

$$Ax - \lambda x = y \tag{3}$$

for some $y \in C[a,b]$. Since $\frac{d}{dt}Ax(t) = x(t)$, we can substitute $z(t) = Ax(t) \in C^1[a,b], \dot{z}(t) = x(t)$ and rewrite (3) as

$$z(t) - \lambda \dot{z}(t) = y(t), z(a) = 0 \Leftrightarrow$$
$$\dot{z}(t) = \frac{1}{\lambda} z(t) - \frac{1}{\lambda} y(t), z(a) = 0$$

By variation of constants formula from ODE, this equation has a unique solution for any fixed $y \in C[a,b]$ given by

$$z(t) = e^{\frac{t}{\lambda}} \frac{1}{\lambda} \int_a^t e^{-\frac{s}{\lambda}} y(s) ds$$

Especially, we obtain $z \equiv 0$ for $y \equiv 0$. Hence, for given $y \in C[a,b]$, we find $x \in C[a,b]$ with $Ax - \lambda x = y$ given by

$$x(t) = \frac{1}{\lambda^2} \int_a^t e^{\frac{(t-s)}{\lambda}} y(s) ds + \frac{1}{\lambda} y(t)$$

which means that A is bijective and hence invertible. This means $\lambda \in \rho(A)$ and the claim is shown.

Alternatively, one can show that

$$\|A^n\| \leq \frac{(b-a)^n}{n!}$$

and use theorem 2.23 to show that

$$r(A) = \lim_{n \to \infty} \|A^n\|^{\frac{1}{n}} = 0.$$

And hence $\sigma(A) \subset \{0\}$. Since $\sigma(A) \neq \emptyset$ by lemma 2.19, we know that $\sigma(A) = \{0\}$.