Cauchy-Transform of Orthogonal Hermite and Laguerre Polynomials in the Complex Plane

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Motivation

Given a matrix model of dimension N that has eigenvalues in the complex plane and can be solved via orthogonal polynomials $p_{N,k}$ on $\mathbb{C}$ with weight function $w_N(z,\bar{z})$. Denote the Cauchy-transform of the polynomials by

$$h_{N,k}(\zeta) := \int_{\mathbb{C}} \frac{p_{N,k}(\bar{z})}{z-\zeta} w_N(z,\bar{z})\,\mathrm{d}^2 z, \quad \deg p_{N,k} = k.$$

We compute the asymptotics of $h_{N,k}(\zeta)$ in the limit $N, k \to \infty$ for the elliptic Ginibre ensemble (Hermite polynomials) and its chiral extension (generalized Laguerre polynomials). This has two applications:

- Universality in the complex plane: Consider the asymptotics of the solution of a Riemann-Hilbert problem (in the case of real eigenvalues) or a $\bar{\partial}$-problem (in the case of complex eigenvalues, see e.g. [1]). However the mathematical theory of the $\bar{\partial}$-problem is not much developed. For some simple matrix models we can calculate the asymptotics of the $\bar{\partial}$-solution by hand, but we need to analyze the above Cauchy-transform for that.
- Quantum chromodynamics: Bosonic partition function for nonzero chemical potential

Outline of Strategy

Idea: Swap $\lim$ with $\int$ and use the asymptotics of the polynomials. **We assume that swapping is allowed, but this still needs to be justified.**

- Find a limiting formula of Mehler-Heine type (subscript MH) for the polynomials, i.e.

$$\lim_{N,k\to\infty} p_{N,k}\left(\frac{z}{N^s}\right) = p_{\mathrm{MH}}(z), \quad s > 0$$

- Rescale the integration variable z and adjust the integration measure:

$$z = \frac{\hat{z}}{N^s} \implies \frac{1}{z-\zeta}\,\mathrm{d}^2 z = \frac{1}{N^s}\frac{1}{\hat{z}-\hat{\zeta}}\,\mathrm{d}^2\hat{z} \quad \text{with } \hat{\zeta} := N^s\zeta$$

i.e. we use the scaling limit at the origin

- Take the pointwise limit of the weight function with rescaled argument

$$w_N(z,\bar{z}) = w_N\left(\frac{\hat{z}}{N^s}, \frac{\bar{\hat{z}}}{N^s}\right) \to w_{\mathrm{MH}}(\hat{z},\bar{\hat{z}})$$

Our Cauchy-transform now looks like this:

$$\lim_{N,k\to\infty} h_{N,k}\left(\frac{\hat{\zeta}}{N^s}\right) = \lim_{N,k\to\infty} \mathcal{C}_{N,k} \int_{\mathbb{C}} \frac{p_{\mathrm{MH}}(\bar{\hat{z}})}{\hat{z}-\hat{\zeta}} w_{\mathrm{MH}}(\hat{z},\bar{\hat{z}})\,\mathrm{d}^2\hat{z} \qquad (1)$$

all constants have been put into $\mathcal{C}_{N,k}$ such that the integral is independent of N and k.

- Under the condition

$$\exists q \in (0,\infty),\ l \in \mathbb{Z}: \quad \lim_{N,k\to\infty} (k - qN - l)\log(N) = 0$$

we can compute $\mathcal{C}_{N,k} \sim \mathcal{C}_N$ for $N, k \to \infty$ using the notion of asymptotic equality.

- Instead of an integral over the complex $\hat{z}$-plane rewrite the integral in (1) as two real integrals using $\hat{z} = x + \mathrm{i}y$ with $x, y \in (-\infty,\infty)$. Calculate one of them via the residue theorem and Jordan's lemma:

$$\int_{-\infty}^{\infty} \frac{f(u)}{u-u_0}\,\mathrm{d}u = 2\pi\mathrm{i}\,\theta(\mathrm{Im}(u_0)) f(u_0), \quad u_0 \in \mathbb{C}\setminus\mathbb{R}$$

where the function $u \mapsto f(u)\mathrm{e}^{-\mathrm{i}au}$, $a > 0$ is bounded in the upper complex halfplane. Here: $f(u) = p_{\mathrm{MH}}(u - \mathrm{i}y) w_{\mathrm{MH}}(u + \mathrm{i}y, u - \mathrm{i}y)$, $u_0 = \hat{\zeta} - \mathrm{i}y$ (for x-integration).

Hermite Polynomials

The *elliptic Ginibre ensemble* consists of matrices X of the form

$$X = \sqrt{1+\tau}\,A + \mathrm{i}\sqrt{1-\tau}\,B, \quad \tau \in [0,1)$$

where A and B are $N \times N$-matrices from the GUE. The parameter τ interpolates between the Ginibre ensemble ($\tau = 0$) and the GUE ($\tau = 1$). When taking the limit $N \to \infty$ we may take $\tau \to 1$ such that $N(1-\tau) = \alpha^2$ stays constant (*weak non-hermiticity limit*). Note that the end result was also obtained in [2] using the same strategy.

- Weight function:

$$w(z,\bar{z}) = \exp\left[-\frac{N}{1-\tau^2}\left(|z|^2 - \frac{\tau}{2}(z^2 + \bar{z}^2)\right)\right]$$

- Orthogonal polynomials:

$$p_{N,k}(z) = \sqrt{\frac{\tau}{2N}}^{\,k} H_k\left(\sqrt{\frac{N}{2\tau}}z\right)$$

where H_k denotes the k-th Hermite polynomial.

- Known Mehler-Heine formula for Hermite polynomials:

$$\lim_{m\to\infty} \frac{(-1)^m\sqrt{m}}{2^{2m(+1)}m!} H_{2m(+1)}\left(\frac{z}{2\sqrt{m}}\right) = \frac{1}{\sqrt{\pi}} \begin{cases} \cos(z) & \text{if } H_{2m} \\ \sin(z) & \text{if } H_{2m+1} \end{cases}$$

- Limit of rescaled weight function:

$$\lim_{N\to\infty} w_N\left(\frac{\hat{z}}{N}, \frac{\bar{\hat{z}}}{N}\right) = \exp\left(-\frac{\mathrm{Im}(\hat{z})^2}{\alpha^2}\right)$$

- Main result (in the weak non-hermiticity limit, see also [2, eq. (110)]):

$$h_{N,k}\left(\frac{\zeta}{N}\right) \sim (-1)^{\lfloor \frac{k}{2} \rfloor}\pi\sqrt{2\pi}\,\alpha\,\mathrm{e}^{q\frac{\alpha^2}{2}}\mathrm{e}^{-\frac{1}{2}qN}\sqrt{q}^{\,qN+l}N^{-1} \begin{cases} -S_-(\sqrt{q}\zeta) & \text{if } k \text{ even} \\ \mathrm{i}S_+(\sqrt{q}\zeta) & \text{if } k \text{ odd} \end{cases}$$

$$S_{\mp}(z) = \frac{1}{2\mathrm{i}}\left[\mathrm{e}^{\mathrm{i}z}\,\mathrm{erfc}\left(\sqrt{q}\alpha - \frac{\mathrm{Im}(z)}{\sqrt{q}\alpha}\right) \mp \mathrm{e}^{-\mathrm{i}z}\,\mathrm{erfc}\left(\sqrt{q}\alpha + \frac{\mathrm{Im}(z)}{\sqrt{q}\alpha}\right)\right]$$

Laguerre Polynomials

The *chiral extension of the elliptic Ginibre ensemble* consists of matrices of the form

$$D = \begin{pmatrix} 0 & \mathrm{i}W_1 + \mu W_2 \\ \mathrm{i}W_1^\dagger + \mu W_2^\dagger & 0 \end{pmatrix}$$

where W_1 and W_2 are complex $N \times (N+\nu)$ Ginibre matrices, $\nu \in \mathbb{N}$. This matrix model can be used as an approximation for the Dirac-operator in QCD with chemical potential $\mu \geq 0$. This approximation is only valid if, among other assumptions, $\mu \to 0$ such that $\mu\sqrt{N} = \hat{\mu}$ stays constant as $N \to \infty$ (*weak non-hermiticity limit*).
Note that the following calculations were already done for $\zeta \in \mathbb{R}$ and $\nu = 0$ in [3], we extend them to $\zeta \in \mathbb{C}$ and $\nu > 0$.

- Weight function:

$$w_N(z,\bar{z}) = |z|^{2(1+\nu)} K_\nu\left(N\frac{1+\mu^2}{2\mu^2}|z|^2\right)\exp\left(-N\frac{1-\mu^2}{2\mu^2}\mathrm{Re}(z^2)\right)$$

where K_ν is the modified Bessel function of the second kind.

- Orthogonal polynomials and Mehler-Heine formula:

$$p_{N,k}(z) = \left(\frac{1-\mu^2}{N}\right)^k k!\, L_k^{(\nu)}\left(-\frac{N}{1-\mu^2}z^2\right)$$

$$\lim_{k\to\infty} k^{-\nu} L_k^{(\nu)}\left(-\frac{z^2}{4k}\right) = 2^\nu z^{-\nu} I_\nu(z)$$

where $L_k^{(\nu)}$ denotes the k-th generalized Laguerre polynomial and I_ν is the modified Bessel function of the first kind.

- Limit of rescaled weight function:

$$w\left(\frac{\hat{z}}{2N}, \frac{\bar{\hat{z}}}{2N}\right) \sim \frac{|\hat{z}|^{2(1+\nu)}}{(2N)^{2(1+\nu)}}\exp\left(-\frac{\mathrm{Re}(\hat{z}^2)}{4\hat{\mu}^2}\right)K_\nu\left(\frac{|\hat{z}|^2}{4\hat{\mu}^2}\right)$$

- We compute the y-integral first, but before using the residue theorem and Jordan's lemma we need to adjust the integration contour to avoid the cut of $K_\nu(x^2+y^2)$:

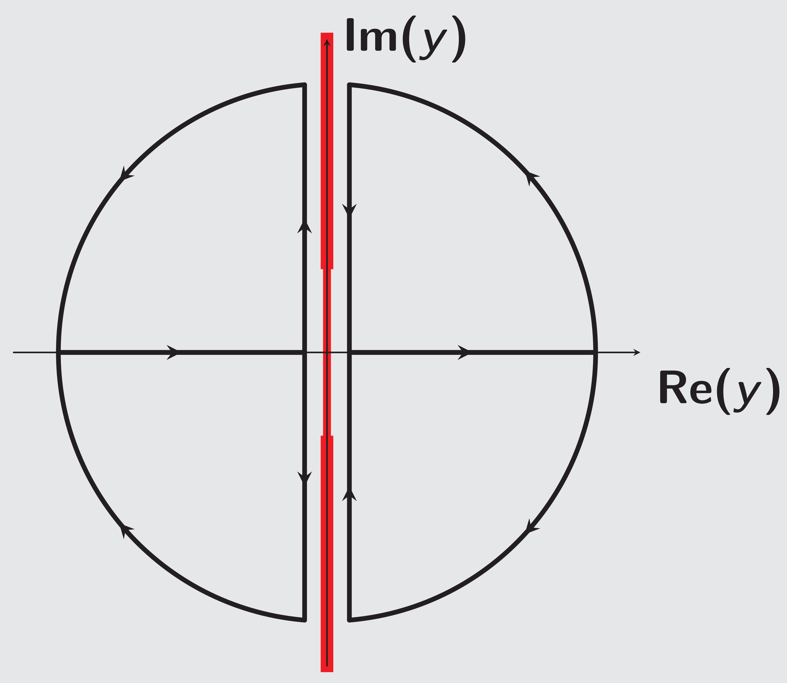

Figure 1: The integration contour (thick black line) is split to avoid the cut of the Bessel-K function (thick red line). We also need to decompose the Bessel-I function to make Jordan's lemma work, the used decomposition involves more Bessel-K functions (with cuts along the thin red line).

Jordan's lemma does not work for $\nu > \frac{3}{2}$, however the end result is still well-defined in this case!

- Main result for $\zeta \in \mathbb{C}\setminus\mathbb{R}$ (for $\zeta \in \mathbb{R}$ use continuity, see also [3, eq. (59)]):

$$h_{N,k}\left(\frac{\zeta}{2N}\right) \sim -\mathrm{sgn}(\mathrm{Im}(\zeta))\,\mathrm{i}\sqrt{\frac{\pi}{2}}\mathrm{e}^{-q\frac{\hat{\mu}^2}{2}-qN}q^{\frac{1}{2}+qN+l}N^{-\frac{3}{2}}\left(\frac{\zeta}{2N}\right)^\nu \times \left[\int_{-\infty}^{\mathrm{Re}(\zeta)} f_-(x)\,\mathrm{d}x + (-1)^\nu\int_{\mathrm{Re}(\zeta)}^{\infty} f_+(x)\,\mathrm{d}x\right]$$

$$f_\pm(x) = (2x-\zeta)\exp\left(-\frac{2x^2 - 2x\zeta + \zeta^2}{4\hat{\mu}^2}\right)K_\nu\left(\frac{(2x-\zeta)\zeta}{4\hat{\mu}^2}\right)K_\nu\big(\pm q(2x-\zeta)\big)$$

Limit to the Real Case

To check the plausibility of our end results we can compare them with the GUE in the Hermite case and the chiral extension of the GUE in the Laguerre case. Denote by g the asymptotic behavior of $h_{N,k}$ and by a superscript whether we use the results for complex or real eigenvalues, then the following diagram should be commutative:

$$\begin{array}{ccc} h_{N,k}^{\mathbb{C}} & \xrightarrow{H_k:\ \tau\to 1,\ L_k^{(\nu)}:\ \mu\to 0} & h_{N,k}^{\mathbb{R}} \\ \Big\downarrow{\scriptstyle N,k\to\infty} & & \Big\downarrow{\scriptstyle N,k\to\infty} \\ g^{\mathbb{C}} & \xrightarrow{H_k:\ \alpha\to 0,\ L_k^{(\nu)}:\ \hat{\mu}\to 0} & g^{\mathbb{R}} \end{array}$$

Taking the results for $g^{\mathbb{R}}$ from [4] we see that in our two cases the limit $g^{\mathbb{C}} \to g^{\mathbb{R}}$ works.

References

[1] Alexander R. Its, Leon A. Takhtajan, arXiv:0708.3867

[2] K. Splittorff, J.J.M. Verbaarschot, M.R. Zirnbauer, Nuclear Physics B, Volume 803, 381-404 (2008)

[3] K. Splittorff, J.J.M. Verbaarschot, Nuclear Physics B, Volume 757, 259-279 (2006)

[4] G. Akemann, Y.V. Fyodorov, Nuclear Physics B, Volume 664, 457-476 (2003)

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