Random dynamical systems Lecture III: Lyapunov exponents and bifurcations

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Joint work with A. Blumenthal, M. Breden, T. S. Doan, C. Kuehn, J. S. W. Lamb, A. Neamtu and M. Rasmussen

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Local bifurcations

Consider the family of **ordinary differential equations** (ODEs)

$$\frac{\mathrm{d}x(t)}{\mathrm{d}t} = f_\alpha(x(t)),\ x(0) \in \mathbb{R}^d,$$

with smooth vector fields f_α, depending on parameter $\alpha \in \mathbb{R}$.

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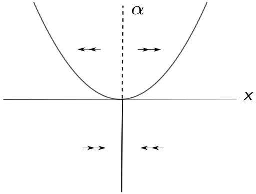
$\rightarrow$ equilibrium changes **stability** and new objects may appear



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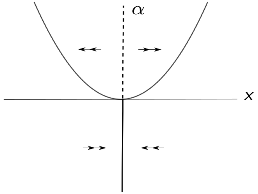

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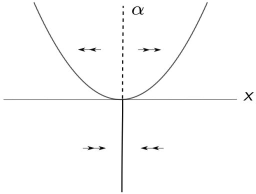

- What if the ODE is replaced by a **stochastic differential equation** (SDE)? (climate science, laser dynamics, etc.)
- What kind of **random phenomena** can we observe and describe, in particular in multiple dimensions?

Bifurcations in random dynamical systems from SDEs

Consider a *stochastic differential equation (SDE)* on $\mathbb{R}^d$

$$\mathrm{d}X_t = f_{\boldsymbol{\alpha}}(X_t)\,\mathrm{d}t + \sigma g(X_t) \circ \mathrm{d}W_t\,, \quad X_0 = x \in \mathbb{R}^d\,,$$

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Question 1: Is the bifurcation still present in the stochastic case, and, if yes, in what sense?

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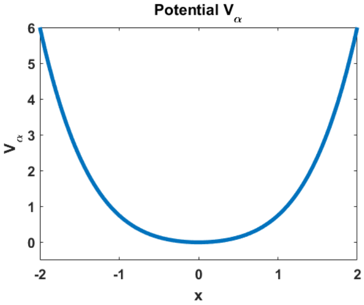
Question 1: Is the bifurcation still present in the stochastic case, and, if yes, in what sense?

Question 2: May noise cause a **new bifurcation scenario** in interaction with other parameters?

Pitchfork bifurcation: $f_\alpha(x) = -\partial_x V_\alpha(x)$ with $V_\alpha = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4$.



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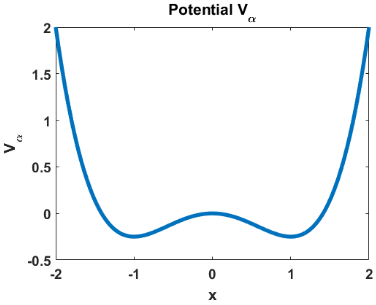
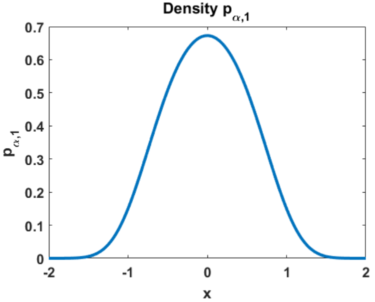


$\alpha = 1$

For $f_\alpha(x) = -\partial_x V_\alpha(x)$ and $g \equiv 1$:

The **stationary** distribution ρ ($P_t^* \rho = \rho$) has the **stationary** density p (solves the stationary Fokker-Planck equation $L^* p = 0$):



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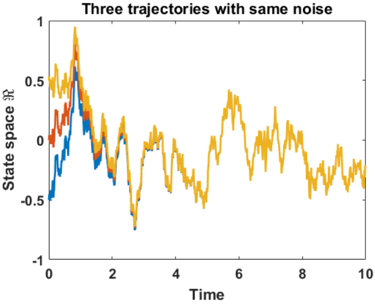
Random dynamical system (θ, φ) as solution of SDE:

For fixed $\omega \in \Omega$ and different x_i, we consider $\varphi(t, \omega, x_i)$, where

$$\varphi(0, \omega, \cdot) = \mathrm{id}, \quad \varphi(t+s, \omega, \cdot) = \varphi(t, \theta_s \omega, \cdot) \circ \varphi(s, \omega, \cdot)\,,$$

and $(\theta_t)_{t \in \mathbb{R}}$ are the time shifts on Ω.



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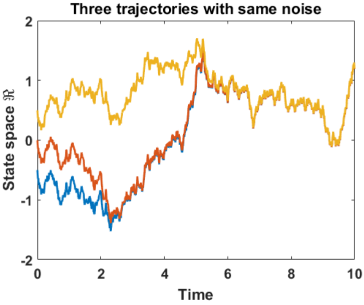

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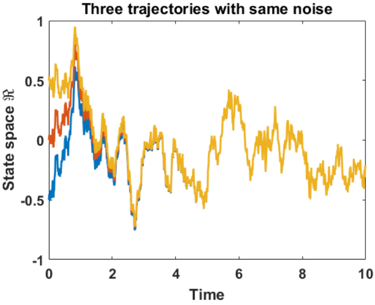

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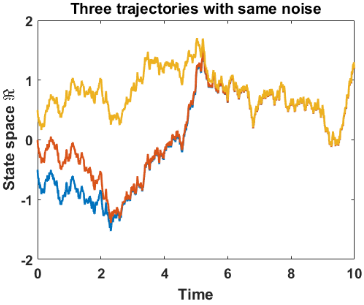

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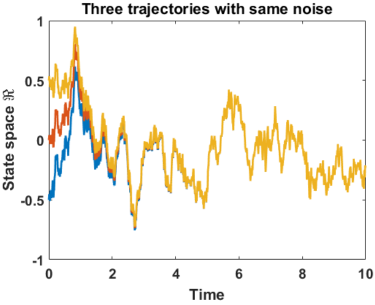

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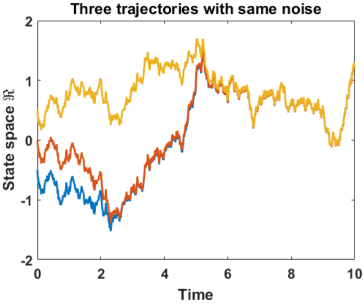

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- **Bifurcation destroyed** by **synchronization** of trajectories $(\lambda_1 < 0)$?
  $\rightarrow$ Not necessarily, consider **finite-time Lyapunov exponents**!

Hopf normal form with additive noise

Hopf-type SDE[1] with **shear** strength $\boldsymbol{b} \in \mathbb{R}$ (phase-amplitude coupling)

$$\begin{aligned} \mathrm{d}x &= (\alpha x - \beta y - (ax - \boldsymbol{b} y)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1\,, \\ \mathrm{d}y &= (\alpha y + \beta x - (\boldsymbol{b} x + ay)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^2\,. \end{aligned}$$

[1] [Wieczorek 2009, DeVille et al. 2011]

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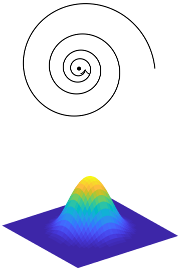
Phase portrait for $\sigma = 0$:

$$p_\alpha(x, y) = \frac{1}{Z} \exp\left(\frac{2\alpha(x^2 + y^2) - a(x^2 + y^2)^2}{2\sigma^2}\right)$$



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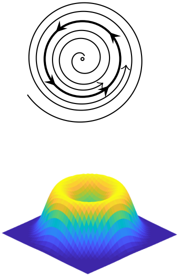

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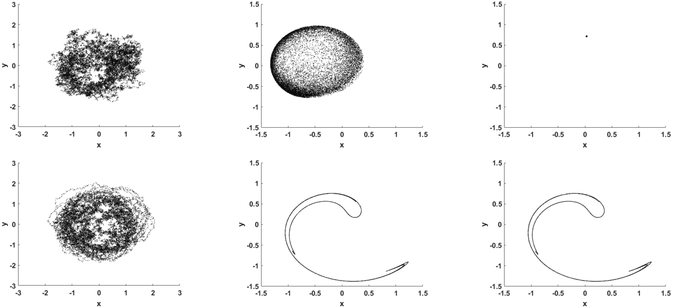

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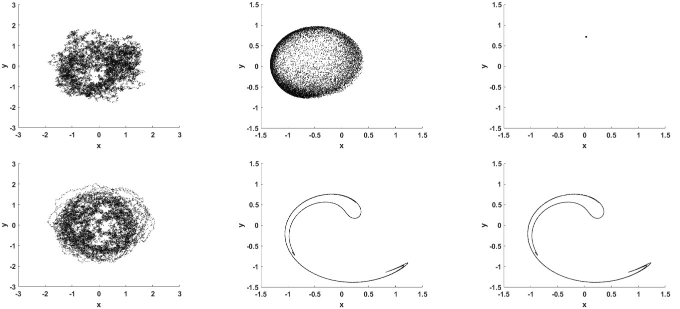

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Crucial quantity is λ_1 from **Lyapunov exponents** $\lambda_1 > \cdots > \lambda_p$ with

$$\lim_{t \to \infty} \frac{1}{t} \ln \|\mathrm{D}_x \varphi(t, \omega, x) v\| \in \{\lambda_i\}_{i=1,\dots,p}, \;\; v \in \mathbb{R}^d \setminus \{0\}.$$

Random attractors

- **Random (pullback) attractors** are sets that satisfy for **almost surely**

$$\lim_{t\to\infty} \operatorname{dist}(\varphi(t, \theta_{-t}\omega, x), A(\omega)) \to 0$$

for all $x \in X$, and for all $t \in \mathbb{T}$

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- For (strongly mixing) Markov RDS with unique invariant **Markov measure**: $A(\omega) = \text{supp}\, \mu_\omega$ is a weak random attractor.

Furstenberg-Khasminskii formula

Let $f \in C^1$ and consider, as in our Hopf example, the SDE

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Then the **finite-time Lyapunov exponents** are

$$\lambda_v(t,\omega,x) = \frac{1}{t}\ln \|\mathrm{D}_x\varphi(t,\omega,x)v\|,$$

where $\mathrm{D}_x\varphi(t,\omega,x)$ solves the linear variational equation

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where $s_t(\omega,x,v) = \frac{\mathrm{D}_x\varphi(t,\omega,x)v}{\|\mathrm{D}_x\varphi(t,\omega,x)v\|}$. By **ergodicity** (and hypoellipticity)

$$\lambda_1 = \int_{\mathbb{R}^d\times\mathbb{S}^1} \langle s, \mathrm{D}_x f(x)s\rangle\,\mathrm{d}\rho(x,s)\,.$$

Synchronisation and bifurcation to chaos in Hopf model

Theorem (Doan/E./Lamb/Rasmussen 2018)

For $|\boldsymbol{b}|$ small, we have $\boldsymbol{\lambda_1} < 0$, and the ***random attractor*** *A is a* ***random equilibrium*** *to which almost all trajectories* ***synchronize****.*

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Chaotic case very difficult due to finding lower bounds for

$$\lambda_1 = \int_{\mathbb{R}^2 \times \mathbb{S}^1} \langle s, \mathrm{D}f_{\alpha,b}(x)s \rangle \, \mathrm{d}\rho_{\alpha,b}(x, y, s) \,,$$

where $\rho_{\alpha,b}(x, y, s)$ solves **multidimensional PDE**.

Simplified model for perturbed limit cycles

The model is given by the SDE (inspired by [Lin/Young 2008])

$$\begin{aligned} \mathrm{d}y_t &= -\alpha y_t \mathrm{d}t + \sigma \sum_{i=1}^{m} f_i(\vartheta) \circ \mathrm{d}W_t^i \,, \\ \mathrm{d}\vartheta_t &= (1 + b y_t)\mathrm{d}t \,, \end{aligned}$$

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- W_t^i denote m independent one-dimensional Brownian motions,
- $\alpha, \sigma, b > 0$ are real parameters,
- we assume that the $f_i : \mathbb{S}^1 \simeq [0, 1) \to \mathbb{R}$ are twice differentiable almost everywhere.

Invariant Measures and random attractor

- For our model, there is an ergodic invariant measure μ with $\mathcal{F}_{-\infty}^{0}$-measurable **disintegrations/sample measures** $(\mu_\omega)_{\omega \in \Omega}$.

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- $A(\omega) := \mathrm{supp}(\mu_\omega)$ is a **random attractor**, as introduced above.

The Furstenberg-Khasminskii formula

The **variational equation** for our model reads

$$\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \sigma \sum_{i=1}^{m} \begin{pmatrix} 0 & f_i'(\vartheta) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^i \,.$$

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Introducing

$$r = \|v\| \quad \text{and} \quad (\cos\phi, \sin\phi) = v/r,$$

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where ρ denotes the **joint invariant measure** of the three variables.

Reduction to one-dimensional problem

Proposition

Consider our model and assume that

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where $p(\phi)$ is the solution of the ***stationary Fokker-Planck equation***

$$\mathcal{L}^* p = 0,$$

associated to ϕ_t.

Main bifucation result

Consider the SDE with $m \geq 2$ and f_i, $i \in \{1, \ldots, m\}$, satisfying the sum condition, as given above.

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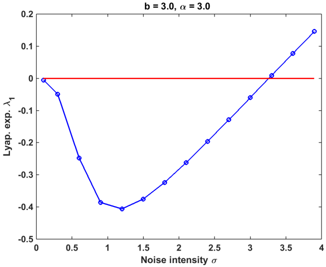
$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \textit{if } 0 < \sigma < \sigma_0(\alpha, b)\,, \\ = 0 & \textit{if } \sigma = \sigma_0(\alpha, b)\,, \\ > 0 & \textit{if } \sigma > \sigma_0(\alpha, b)\,. \end{cases}$$



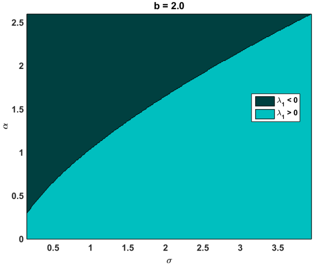
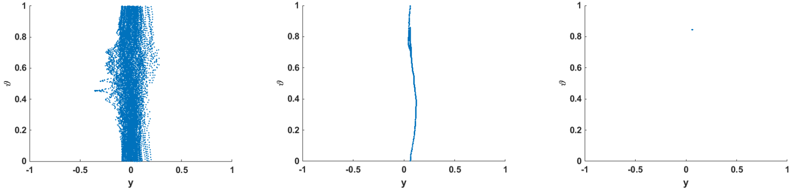


Figure: In Figure (a) the first Lyapunov exponent λ_1 is shown as a function of σ for fixed b and α. Figure (b) shows the areas of positive and negative λ_1 in the (σ, α)-parameter space being separated by the curve $\{(\sigma_0(\alpha, 2), \alpha)\}$.

Synchronization of trajectories

If $0 < \sigma < \sigma_-(\alpha, b)$, the random attractor is an **attracting random equilibrium**:

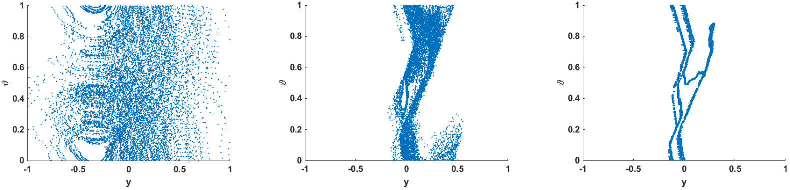


$t = 0$ $t = 10$ $t = 50$

Figure: *Approximating the support of* $\mu_\omega = \lim_{t\to\infty} \varphi(t, \theta_{-t}\omega)\rho$ *for fixed* ω. *The parameters are* $\sigma = 0.5, \alpha = 1.5, b = 3$ *such that* $\lambda_1 < 0$.

Chaotic attractor

If $\sigma > \sigma_+(\alpha, b)$ the random attractor is a **random strange attractor** (and not an attracting random equilibrium):



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Observation: Stability and bifurcation phenomena **local** but noise **global**!

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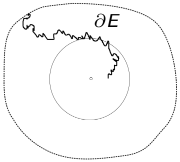
$$T := \inf\{t \geq 0, X_t \in \partial E\}$$

and let $X_t = X_T$ for all $t \geq T$.



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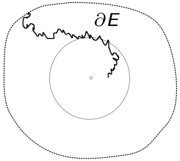

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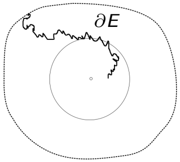

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The probability measure m is a *quasi-ergodic distribution (QED)* if for every bounded and measurable function h and every $x \in E$

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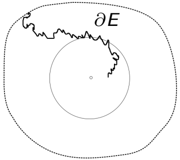

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- Typically, the **QED** $\mathrm{d}m = \eta \, \mathrm{d}\nu$, where ν (**QSD**) and η can be found as eigenfunctions of **Kolmogorov** operators L^* and L.

Conditioned Lyapunov exponent

Theorem C (E./Lamb/Rasmussen, Trans. Amer. Math. Soc. 2019)

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- We can see $\lambda_c > 0$ as a measure of **chaos**.

Application to stochastic Hopf bifurcation

In polar coordinates (r, ψ), the **FK functional** $\langle s, Df(x)s \rangle$ becomes

$$e(r, \psi) = \alpha - 2ar^2 + r^2\sqrt{a^2 + \mathbf{b}^2}\sin\psi.$$

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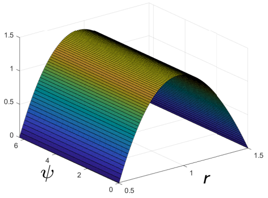

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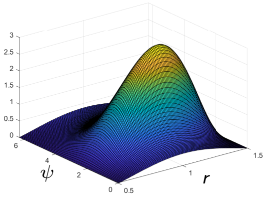

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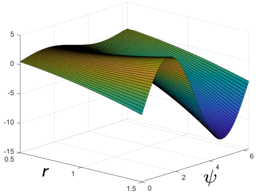

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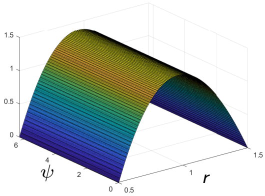

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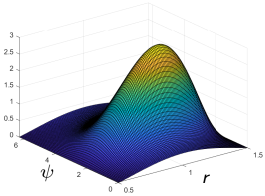

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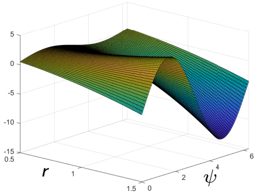

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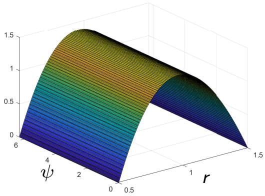

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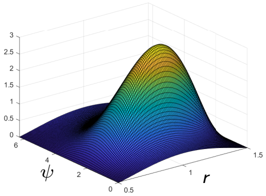

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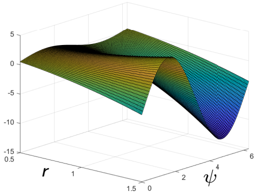

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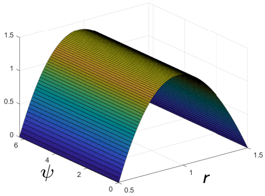

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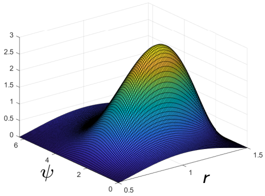

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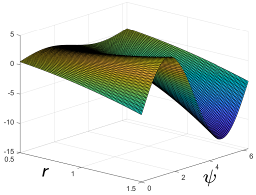

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- **Rigorously** compute η (eigenvector of L) and ϕ (eigenvector of L^*).
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- Then we can prove whether λ_c is **positive** or **negative**.

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Using MATLAB + INTLAB for the interval arithmetic computations:

Theorem D (Breden/E. 2021)

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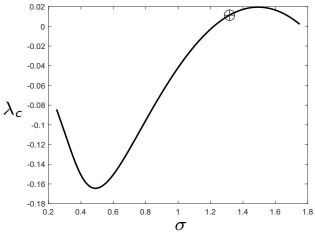

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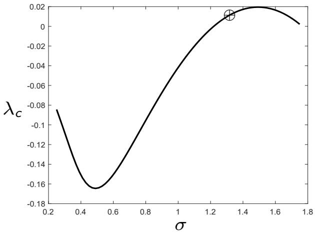

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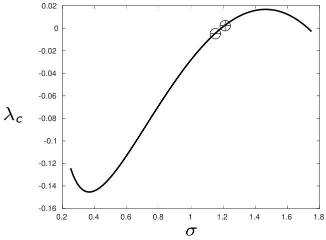

Figure: $[r_{min}, r_{max}] = [0.75, 1.25]$

Conditioned Lyapunov exponent λ_c as a function of σ for fixed parameter values $b = 3.6, \alpha = a = 1$, on annuli of different lengths $[r_{\min}, r_{\max}]$.

Outlook

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- Conditioned Lyapunov **spectrum**

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References

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Thank you very much for your attention!

Details on computer-assisted proof: operators L and L^*

Setting

$$f(r) = \alpha r - a r^3 + \frac{\sigma^2}{2r}, \qquad g(r,\psi) = 2r^2\left(\mathbf{b} + \sqrt{a^2+\mathbf{b}^2}\cos\psi\right),$$

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the Kolmogorov operators take the form

$$Lu = \frac{\sigma^2}{2}\left(\frac{\partial^2 u}{\partial r^2} + \frac{\mathbf{4}}{\mathbf{r^2}}\frac{\partial^2 u}{\partial \psi^2}\right) + f(r)\frac{\partial u}{\partial r} + g(r,\psi)\frac{\partial u}{\partial \psi},$$

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- Our main task is then to prove that T is a **contraction** on a (small and explicit) neighborhood of $(\bar{u}, \bar{\lambda})$.

A theorem[2] in the spirit of Newton-Kantorovich

Theorem E

Let $\mathcal{X} = H^2_B(\Omega) \times \mathbb{C}$, $\mathcal{Y} = L^2(\Omega) \times \mathbb{C}$ *and* $\varepsilon, \kappa, \gamma > 0$ *such that*

$$\begin{aligned} \left\| F(\bar{u}, \bar{\lambda}) \right\|_{\mathcal{Y}} &\leq \varepsilon \\ \left\| F'(\bar{u}, \bar{\lambda})^{-1} \right\|_{\mathcal{Y} \to \mathcal{X}} &\leq \kappa \\ \left\| F'(u, l) - F'(\bar{u}, \bar{\lambda}) \right\|_{\mathcal{X} \to \mathcal{Y}} &\leq \gamma \left\| (u, l) - (\bar{u}, \bar{\lambda}) \right\|_{\mathcal{X}} \qquad \forall\, (u, l) \in \mathcal{X}. \end{aligned}$$

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If

$$\varepsilon < \frac{1}{2\kappa^2\gamma},$$

then F *has a unique zero* $(u, l) \in \mathcal{X}$ *satisfying* $\left\| (u, l) - (\bar{u}, \bar{\lambda}) \right\|_{\mathcal{X}} \leq r$*, where*

$$r = \frac{1 - \sqrt{1 - 2\kappa^2\gamma\varepsilon}}{\kappa\gamma}.$$

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The main estimate

$$Lu = \frac{\sigma^2}{2}\left(\frac{\partial^2 u}{\partial r^2} + \frac{4}{r^2}\frac{\partial^2 u}{\partial \psi^2}\right) + f\frac{\partial u}{\partial r} + g\frac{\partial u}{\partial \psi}, \quad F(u,\lambda) = \begin{pmatrix} Lu - \lambda u \\ \langle u, \bar{u}\rangle - 1 \end{pmatrix}$$

- **Most challenging part** of the validation: find $\kappa > 0$ such that

$$\|F'(\bar{u},\bar{\lambda})^{-1}\|_{\mathcal{Y}\to\mathcal{X}} \leq \kappa,$$

or equivalently

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- We then combine this with **a priori** estimates of the form

  $\|[F'(\bar{u},\bar{\lambda})]\,(u,\lambda)\|_{\mathcal{Y}} \geq c_1\|\nabla u\|_{L^2}, \quad \|[F'(\bar{u},\bar{\lambda})]\,(u,\lambda)\|_{\mathcal{Y}} \geq c_2\|\Delta u\|_{L^2},$

  in order to get $1/\kappa$.