2. Linear RDS, Multiplicative Ergodic Theorem and Lyapunov exponents

In the following, we focus on $X = \mathbb{R}^d$.

More generally, X may be a smooth manifold (see Arnold '98) or a Banach space (see González-Tokman '18).

2.1 Linear RDS

Def.: (θ, φ) is linear if the map

$$\varphi(t,\omega) : \mathbb{R}^d \to \mathbb{R}^d, \quad x \mapsto \varphi(t,\omega,x)$$

is linear for all $(t,\omega) \in \mathbb{T} \times \Omega$.

Set $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^{d \times d}$, $\quad \Phi(t,\omega)x := \varphi(t,\omega,x)$.

We will investigate the Lyapunov exponents

$$\lambda(\omega,x) = \limsup_{t\to\infty} \frac{1}{t} \log \| \Phi(t,\omega)x \|, \quad \omega \in \Omega, \; 0 \neq x \in \mathbb{R}^d$$

Example of linear RDS:

Generally: (θ, φ) RDS with invariant measure μ

$$\to \boxed{D_x \varphi(t,\omega,x) := \frac{\partial \varphi(t,\omega,x)}{\partial x} \in \mathbb{R}^{d\times d}}$$

Differentiate and apply chain rule to

$$\varphi(t+s,\omega,x) = \varphi(t, \theta_s\omega, \varphi(s,\omega,x)) \quad \to$$

$\longrightarrow \quad D_x \varphi(t+s, \omega, x) = D_x \varphi(t, \bar{\Theta}_s(\omega, x))\, D_x \varphi(s, \omega, x),$

i.e. $D_x\varphi$ cocycle w.r.t. skew-product $(\bar{\Theta}_t)_{t\in\mathbb{T}}$,

$\longrightarrow \quad (\bar{\Theta}, D_x\varphi)$ is an RDS.

Specifically: Recall RDS (θ, φ) from

(SDE) $$\boxed{dX_t = f(X_t)\,dt + \sum_{j=1}^{m} g_j(X_t) \circ dW_t^j}$$

$D_x \varphi(t, \omega, x)\, v_0$, $v_0 \in \mathbb{R}^d$, is the solution to the variational equation

$$\boxed{\begin{aligned} dv &= D_x f(\varphi(t,\omega,x))\, v\, dt + \sum_{j=1}^{m} D_x g_j(\varphi(t,\omega,x)) \circ dW_t^j, \\ & \qquad v(0) = v_0 \in \mathbb{R}^d. \end{aligned}}$$

2.2 Kingman's SET and Furstenberg-Kesten

- $\mathbb{T} \in \{\mathbb{Z}, \mathbb{Z}_0^+\}$, $\theta: \Omega \to \Omega$ meas.-pres. on $(\Omega, \mathcal{F}, \mathbb{P})$,
- $\phi: \Omega \to [-\infty, \infty]$ measurable;

  $\varphi^+(\omega) = \max\{0, \varphi(\omega)\}$, $\quad \varphi^-(\omega) = \max\{0, -\varphi(\omega)\}$
- φ is invariant if $\varphi(\theta\omega) = \varphi(\omega)$ almost surely

Def.: A sequence $\varphi_n : \Omega \to [-\infty, \infty]$ of meas. maps is subadditive, w.r.t. θ, if

$$\varphi_{n+m} \le \varphi_m + \varphi_n \circ \underbrace{\theta_m}_{:=\theta^m} \quad \text{for all } m, n \ge 1.$$

Example: $A : \Omega \to \mathbb{R}^{d \times d}$, $\phi_n(\omega) = \log \| A^n(\omega) \|$,

where again $A^n(\omega) = A(\theta_{n-1}\omega) \cdots A(\theta\omega) A(\omega)$

(matrix cocycle)

ϕ_n subadditive by sublinearity of matrix norm products.

Theorem [Kingman's Subadditive Ergodic Thm]

$\phi_n : \Omega \to [-\infty, \infty)$, $n \geq 1$, subadditive s.t. $\phi_1^+ \in L^1(\mathbb{P})$. Then we have:

- $\boxed{\frac{\phi_n}{n} \to \phi}$ $\mathbb{P}$-a.s., $\phi : \Omega \to [-\infty, \infty)$ invariant

  (ϕ constant if $\mathbb{P}$ is ergodic for θ)

- $\phi^+ \in L^1(\mathbb{P})$ and $\int \phi \, d\mathbb{P} \in [-\infty, \infty)$.

Proof: See Lecture notes. $\square$

Corollary [Birkhoff's Ergodic Thm]

$\phi : \Omega \to \mathbb{R}$ $\mathbb{P}$-integrable:

- $$\boxed{\lim_{n \to \infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\theta_j \omega) = \tilde{\phi}(\omega) \quad \mathbb{P}\text{-a.s.},}$$

  where $\tilde{\phi}$ invariant and $\int \tilde{\phi} \, d\mathbb{P} = \int \phi \, d\mathbb{P}$.

- If $\mathbb{P}$ ergodic, $\tilde{\phi} = \int \phi \, d\mathbb{P}$ a.s..

Proof: Set $\phi_n = \sum_{j=0}^{n-1} \phi(\theta_j \omega)$: (sub)-additive! $\square$

(See Viana '14)

- Consider linear RDS (θ, Φ), where
$$\Phi(n,\omega) = A^n(\omega) = A(\theta_{n-1}\omega) \cdots A(\omega), \qquad A: \Omega \to \mathbb{R}^{d\times d}$$
- We will write $\log^+(x) = \max\{0, \log(x)\}$, $x \in (0,\infty)$.

Theorem [Furstenberg-Kesten]

(IA) $\log^+ \|A^{\pm 1}\| \in L^1(\mathbb{P})$:

⊙ Then the extremal Lyapunov exponents
$$\lambda_+(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)\|, \quad \lambda_-(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|(A^n(\omega))^{-1}\|^{-1}$$
exist for almost all $\omega \in \Omega$.

⊙ $\lambda_\pm$ invariant and $\mathbb{P}$-integrable, i.e., if $\mathbb{P}$ ergodic, $\lambda_\pm$ are constants.

Proof:
- $\varphi_n = \log \|A^n(\omega)\|$, $\psi_n = \log \|(A^n(\omega))^{-1}\|$
- By (IA): $\varphi_1^+, \psi_1^+ \in L^1(\mathbb{P})$ and, hence,
$$\varphi_1(\omega), \psi_1(\omega) \in [-\infty, \infty) \text{ a.s.}$$
- φ_n, ψ_n are subadditive (see above)

$\to$ Statements follow by Kingman's SET. $\square$

2.3 Multiplicative Ergodic theorems

Linear RDS (θ, Φ), on $(\Omega, \mathcal{F}, \mathbb{P})$, $\mathbb{R}^d$:

Discrete time: $\Phi(1,\omega) = A(\omega)$ (generating matrix)

Cont. time, main example: $\Phi(t,\omega) = \partial_x \varphi(t,\omega,x)$, where $\varphi(t,\omega,x)$ solves (SDE) $\to$ Φ is cocycle over

$\Theta_t : \Omega := \tilde{\Omega} \times \mathbb{R}^d \to \Omega$, $\Theta_t(\omega) = (\tilde{\Theta}_t \tilde{\omega}, \varphi(t,\tilde{\omega},x))$

- $\tilde{\Theta}_t$ time shift on Wiener space $\tilde{\Omega}$
- Θ_t has invariant Markov measure μ

(corresponding with stationary ρ)

Def: A flag (or filtration) is a family $W^1 \supseteq \dots \supseteq W^k \supset \{0\}$ of subspaces of $\mathbb{R}^d$.

We first state the MET for Θ not necess. invertible.

Theorem [Oseledets' MET in one-sided time]

(A) $\mathbb{T} = \mathbb{Z}_0^+$ and $\log^+ \| A^{\pm 1} \| \in L^1(\mathbb{P})$ (ICI)

Then for a.a. $\omega \in \Omega$, there are $k = k(\omega)$ numbers $\lambda_1(\omega) > \dots > \lambda_k(\omega)$ (Lyapunov exponents) and a flag $\mathbb{R}^d = V_\omega^1 \supsetneq \dots \supsetneq V_\omega^k \supsetneq \{0\}$ s.t. for $i = 1,\dots,k$:

a) $k(\Theta\omega) = k(\omega)$, $\lambda_i(\Theta\omega) = \lambda_i(\omega)$, $A(\omega) V_\omega^i = V_{\Theta\omega}^i$,

b) $\omega \mapsto k(\omega)$, $\omega \mapsto \lambda_i(\omega)$, $\omega \mapsto V_\omega^i$ are measurable

c) $\lim_{n\to\infty} \frac{1}{n} \log \| A^n(\omega) v \| = \lambda_i(\omega)$, $v \in V_\omega^i \setminus V_\omega^{i+1}$

$(V_\omega^{k+1} = \{0\})$

If $\mathbb{P}$ is ergodic:

$k(\omega)$, $\lambda_i(\omega)$, $\dim V_\omega^i$ are constant almost surely!

(B) $\mathbb{T} = \mathbb{R}_0^+$ and

$$\sup_{0 \le t \le 1} \log^+ \| \Phi(t,\cdot)^{\pm 1} \| \in L^1(\mathbb{P}) \qquad \text{(IC II)}$$

then all statements in (A) hold if

$n, \theta, A^n(\omega)$ are replaced by $t, \theta_t, \Phi(t,\omega)$.

Remarks:

(i) $\dim V_\omega^i - \dim V_\omega^{i+1}$ multiplicity of λ_i

(ii) Lyapunov spectrum is set of all Lyap. exponents

(iii) There are proofs that do not require invertibility of $A(\omega)$ or $\Phi(t,\cdot)$

(iii) $\lambda_1(\omega) = \lambda_+(\omega)$, $\lambda_{k(\omega)} = \lambda_-(\omega)$ (see Furstenberg-Kesten)

(iv) $\sum_i \lambda_i(\omega) = \lim_{t \to \infty} \frac{1}{t} \log |\det(\Phi(t,\omega))|$ (Sum of LEs)

Theorem [Oseledets' MET in two-sided time]

Situation as before + $\theta_{(t)} : \Omega \to \Omega$ invertible:

(A) $\mathbb{T} = \mathbb{Z}_0^+$ and (IC I):

For a.a. $\omega \in \Omega$, we have the Oseledets splitting

$$\mathbb{R}^d = E^1(\omega) \oplus \dots \oplus E^k(\omega) \text{ s.t. } \forall i = 1, \dots, k:$$

(a) $A(\omega) E^i(\omega) = E^i(\theta\omega)$, $\quad V_\omega^i = \bigoplus_{j=i}^{k} E^j(\omega)$

(b) $\lim_{n \to \pm\infty} \frac{1}{n} \log \| A^n(\omega) v \| = \lambda_i(\omega)$

for all $v \in E^i(\omega) \setminus \{0\}$

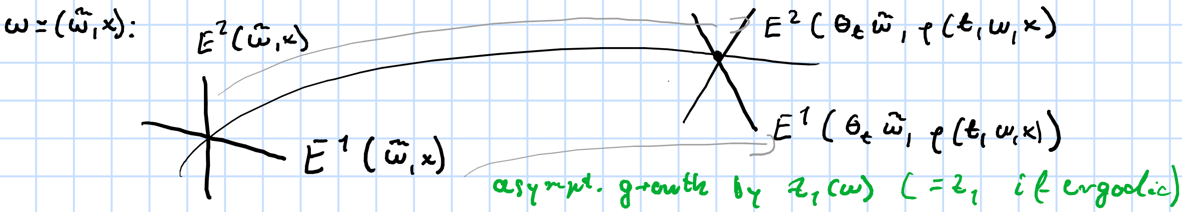
(B) $\mathbb{T} = \mathbb{R}_0^+$, (IC II): Replace $n, \theta, A^n(\omega)$ by $t, \theta_t, \Phi(t,\omega)$.

Remarks:

(i) Multiplicity of $\lambda_i(\omega)$ is $\dim E^i(\omega) = \dim V_\omega^i - \dim V_\omega^{i+1}$

(ii) Lyapunov spectrum simple iff $\dim E^i(\omega) = 1 \quad \forall i$

Sketch of Oseledets splitting in 2d:



Comments on proofs of METs:

1) $\lambda(\omega, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|f^n(\omega) v\|$ (LE)

$\rightarrow$ $q(\omega) = \# L \; (:= \{\lambda(\omega, v) : v \in \mathbb{R}^d \setminus \{0\}\})$

(Observe: $\lambda(\omega, v+v') = \max\{\lambda(\omega,v), \lambda(\omega,v')\}$, if $v+v' \neq 0$)

$\rightarrow$ flag $V_\omega^i = \{v \in \mathbb{R}^d \setminus \{0\} : \lambda(\omega, v) \leq \lambda_i(\omega)\} \cup \{0\}$

$\rightarrow$ $\lambda(\omega, v) = \lambda_i(\omega)$ for all $v \in V_\omega^i \setminus V_\omega^{i+1}$

Invariance of $k(\omega)$, $\lambda_i(\omega)$, V_ω^i easy to observe

2) Measurability: technical proof.

3) Hard part: Replace $\limsup_{n\to\infty}$ in (LE) by $\lim_{n\to\infty}$! We show this for $V_\omega := V_\omega^k$ and then for all V_ω^i by induction, using $\mathbb{R}^d = V_\omega \oplus V_\omega^\perp$.

4) Transition to continuous time, by control of (IC II)

5) Two-sided time: flag -> direct sum decomposition via graph transform techniques (There is also a proof via exterior powers)

2.4. Examples of computing λ_1

2.4.1 Herman's formula

Recall cocycle $\Phi(n,\omega) = A^n(\omega) = A(\theta_{n-1}\omega)\cdots A(\omega)$

for $A(\omega) = \begin{pmatrix} \sigma & 0 \\ 0 & \sigma^{-1} \end{pmatrix} \begin{pmatrix} \cos 2\pi\omega & -\sin 2\pi\omega \\ \sin 2\pi\omega & \cos 2\pi\omega \end{pmatrix}$,

where $\theta : S^1 \to S^1$ is an irrational rotation.

(Viana '14) • $\lambda_1 = \lambda_+ \geq \log \frac{\sigma + \sigma^{-1}}{2}$

> 0 if $\sigma \neq 1$

$\Rightarrow E_1(\omega)$ unstable Oseledets space

• $\lambda_2 = -\lambda_1 \leq -\log \frac{\sigma+\sigma^{-1}}{2} < 0$ if $\sigma \neq 1$

$\Rightarrow E_2(\omega)$ stable Oseledets space

2.4.2 One-dimensional SDE

(SDE1d) $dX_t = f(X_t)\,dt + \sigma\, dW_t$, $f \in C^1(\mathbb{R})$, $\sigma > 0$.

Define $F(x) = \int_0^x f(u)\,du$:

Stationary density $p(x) = c \exp\left(\frac{2F(x)}{\sigma^2}\right)$, $c > 0$.

($L^* p = 0$) (assuming integrability)

$\to$ RDS (θ, φ) with unique ergodic, invariant Markov measure μ (Correspondence Theorem)

$D_x \varphi(t,\omega,x)$ solves variational equation

$$\dot v_t = f'(\varphi(t,\omega,x))\, v_t \,, \quad v_0 \neq 0$$

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \log |D_x \varphi(t,\omega,x)\, v_0|$$

$$= \lim_{t\to\infty} \frac{1}{t} \Big(\log |v_0| + \int_0^t f'(\varphi(s,\omega,x))\, ds \Big)$$

$$\overset{\text{ergodicity}}{=} \int_\Omega \int_{\mathbb{R}} f'(x)\, d\mu_\omega(x)\, d\mathbb{P}(x)$$

$$\overset{\text{Correspondence}}{=} \int_{\mathbb{R}} f'(x)\, \mu(x)\, dx = - \int_{\mathbb{R}} f(x)\, \mu'(x)\, dx$$

$$= -c \int_{\mathbb{R}} f(x) \Big(\frac{2 f(x)}{\sigma^2} \exp(2F(x)/\sigma^2) \Big) dx$$

$$= -\frac{2c}{\sigma^2} \int_{\mathbb{R}} f(x)^2 \exp(2F(x)/\sigma^2)\, dx \boxed{< 0.}$$

$$\Rightarrow \quad E_1(\omega) = \underset{\text{stable}}{E_s}(\omega) = \mathbb{R} \quad \text{for all } \omega.$$

2.4.3 Furstenberg-Khasminskii (FK) formula

$$d\gamma_t = A_0 \gamma_t\, dt + \sum_{j=1}^m A_j \gamma_t \circ dW_t^j \,,$$
$$\gamma_0 = v \in \mathbb{R}^d,$$

where $A_0, \dots, A_m \in \mathbb{R}^{d\times d}$.

Polar coordinates: $r_t = \|\gamma_t\|$, $s_t = \dfrac{\gamma_t}{r_t}$.

Set $g_A(s) = As - \langle s, As \rangle s$, $s \in S^{d-1}$.

Then (s_t, r_t)-process given by

$$\begin{cases} ds_t = g_{A_0}(s_t)\,dt + \sum_{j=1}^m g_{A_j}(s_t) \circ dW_t^j \\ dr_t = \langle s_t, A_0 s_t \rangle r_t\,dt + \sum_{j=1}^m \langle s_t, A_j s_t \rangle r_t \circ dW_t^j \end{cases}$$

Crucial: Solvability of $L^* p = 0$ (stationary FP-Equ.)

for $L = g_{A_0} + \frac{1}{2} \sum_{j=1}^m g_{A_j}^2$

(generator of semigroup P_t)

$\rightarrow$ can be obtained from Hörmander's hypoellipticity condition on g_{A_j}, $j=0,\dots,m$.

In this case: Via ergodicity one gets

$$\boxed{\lambda_1 = \int_{S^{d-1}} \Big[\langle s, A_0 s \rangle + \sum_{j=1}^m \frac{1}{2} \langle (A_j + A_j^*) s, A_j s \rangle - \langle s, A_j s \rangle^2 \Big] \cdot p(s)\,ds}$$

Note: For our general (SDE),

$A_0 = D_x f(\underbrace{\varphi(t,\omega,x)}_{\text{solves (SDE)}})$, $A_j = D_x g_j(\varphi(t,\omega,x))$

$\rightarrow$ FK-formula for joint stationary density of projective bundle process

$(\varphi(t,\omega,x), s(\varphi(t,\omega,x)))$

$\rightarrow$ see final Lecture for examples.