IRTG Bielefeld–Seoul Winter School - Stochastic Dynamics

Metastable dynamics of Markov processes

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Based on joint works with Manon Baudel, Giacomo Di Gesù, Bastien Fernandez, Barbara Gentz, Damien Landon and Hendrik Weber






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2. Continuous-space Markov chains and SDEs

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = \int_A k_\sigma(x,y)\,\mathrm{d}y$$

Deterministic Poincaré maps

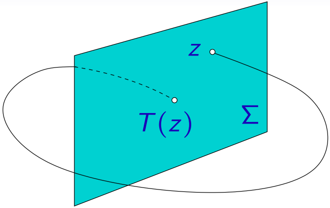
ODE $\dot{z} = f(z)$ $z \in \mathbb{R}^n$

Flow: $z_t = \varphi_t(z_0)$

$\Sigma \subset \mathbb{R}^n$: $(n-1)$-dimensional manifold

Poincaré map (or first-return map):

$T : \Sigma \to \Sigma$



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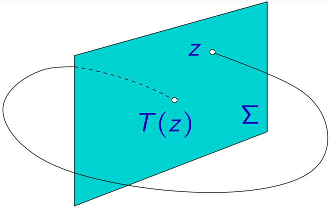

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$T : \Sigma \to \Sigma$

$T(z) = \varphi_\tau(z)$ where $\tau = \inf\{t > 0 : \varphi_t(z) \in \Sigma\}$

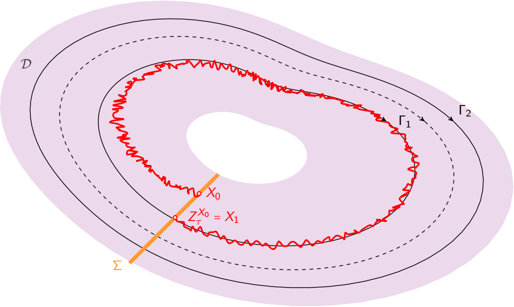
Benefits:

1. Dimension reduction: T is an $(n-1)$-dimensional map
2. Stability of periodic orbits: no neutral direction
3. Bifurcations of periodic orbits easier to study
(period doubling, Hopf, . . .)

Question: how about SDEs $\quad \mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$?

Random Poincaré maps

$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t \quad \Rightarrow \quad$ Sample path $(Z_t^{z_0}(\omega))_{t\geqslant 0}$



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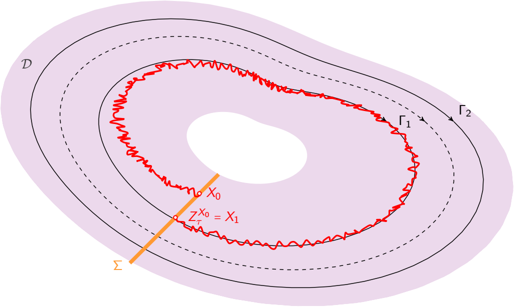

⚠ $z_0 = X_0 \in \Sigma \quad \Rightarrow \quad \inf\{t > 0 \colon Z_t^{X_0} \in \Sigma\} = 0$

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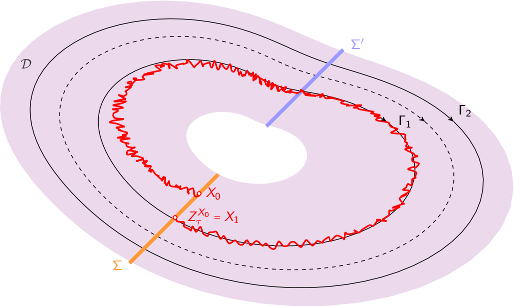

⚠ $z_0 = X_0 \in \Sigma \quad \Rightarrow \quad \inf\{t > 0 \colon Z_t^{X_0} \in \Sigma\} = 0$

Solution: $\tau_0 = 0$, $\tau'_{n+1} = \inf\{t > \tau_n \colon Z_t^{X_0} \in \Sigma'\}$

$\tau_{n+1} = \inf\{t > \tau'_{n+1} \colon Z_t^{X_0} \in \Sigma\}$

$X_n = Z_{\tau_n}^{X_0} \in \Sigma \quad \Rightarrow \quad (X_n)_{n\geqslant 0}$ is a Markov chain $\quad K(x,A) := \mathbb{P}^x\{X_1 \in A\}$

$(X_n, \omega) \mapsto X_{n+1}$: random Poincaré map

[J. Weiss, E. Knobloch, 1990], [P. Hitczenko, G. Medvedev, 2009]

Continuous-space Markov chains

$(X_n)_{n\in\mathbb{N}_0}$ Markov chain in $\mathcal{X} \subset \mathbb{R}^d$ with kernel K_σ:

$$\mathbb{P}\{X_{n+1} \in A | X_n = x\} = K_\sigma(x, A) = \int_A K_\sigma(x, \mathrm{d}y)$$

- $K_0(x, A) = 1_{\{\Pi(x)\in A\}}$ defined by deterministic map $\Pi : \mathcal{X} \to \mathcal{X}$
- For $\sigma > 0$, K_σ admits continuous density k_σ

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Example 1: Randomly perturbed map

$$X_{n+1}=\Pi(X_n)+\sigma\xi_{n+1}$$

$(\xi_n)_{n\geqslant1}$ i.i.d. r.v. with density (e.g. $\sigma\xi_n$ Gaussian of variance σ^2)

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Example 2: Random Poincaré map

SDE

$$\mathrm{d}z_t = f(z_t)\,\mathrm{d}t + \sigma g(z_t)\,\mathrm{d}W_t$$

X_n suitably defined location of nth return to surface of section $\Sigma \subset \mathcal{X}$

Assumptions

Assumption 1: Deterministic dynamics

$\Pi : \mathcal{X} \to \mathcal{X}$ admits positively invariant compact set $\mathcal{X}_0 \subset \mathcal{X}$, finitely many limit sets in $\mathcal{X}_0$, all hyperbolic fixed points, N of which are stable

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Assumption 2: Large-deviation principle

K_σ satisfies LDP with good rate function I $(K_\sigma(x, A) \sim e^{-\inf_A I(x,\cdot)/\sigma^2})$
$I(x, y) = 0 \Leftrightarrow y = \Pi(x)$

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Assumption 4: Uniform positivity (Doeblin-type condition)

$\forall x_i^\star$ stable fixed point, $\exists B_i$ nbh of $x_i^\star$ s.t. $k_i = {}_{B_1\cup\dots\cup B_i} k_{B_i}$ satisfies

$$\sup_{x\in B_i} k_i^n(x,y) \leqslant L \inf_{x\in B_i} k_i^n(x,y) \quad \forall y \in B_i \qquad \text{for some } L \in (1,2),\ n(\sigma) \in \mathbb{N}$$

Metastable hierarchy (SDE case)

Freidlin–Wentzell theory:

Rate function: $I_{[0,T]}(\gamma) = \dfrac{1}{2}\displaystyle\int_0^T (\dot\gamma_s - f(\gamma_s))^{\mathsf{T}} [gg^{\mathsf{T}}(\gamma_s)]^{-1} (\dot\gamma_s - f(\gamma_s))\, \mathrm{d}s$

Large-deviation principle: $\mathbb{P}\left\{(z_t)_{0\leqslant t\leqslant T} \in \Lambda\right\} \simeq \mathrm{e}^{-\inf_{\gamma\in\Lambda} I_{[0,T]}(\gamma)/\sigma^2}$

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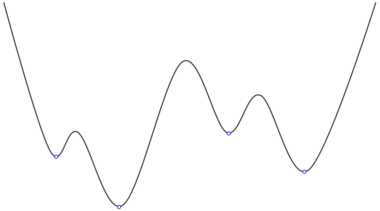
Assumption 5: Metastable hierarchy

$\exists\theta > 0$ s.t. $\forall\ 2 \leqslant k \leqslant N$

$$\min_{\ell<k} H(k,\ell) \leqslant \min_{\substack{i<k\\ j\neq i}} H(i,j) - \theta$$



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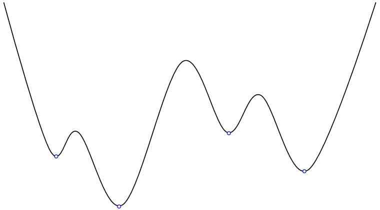

Remark: Using Doob's h-transform, one may replace Assumption 3 by

Assumption 3': Confinement

$\exists\theta' > 0$ such that $\min_i H(i,\partial\mathcal{D}) \geqslant \max_{i\neq j} H(i,j) + \theta'$

Main result

Theorem [Baudel & B, 2017]

- Non-degenerate case ($x_1^\star, \dots, x_N^\star$ in metastable order)
  - Eigenvalues of K_σ:

$$\begin{aligned} \lambda_0 &= 1 \\ \lambda_k &= 1 - \mathbb{P}^{\mathring{\pi}_0^{k+1}}\{\tau^+_{B_1\cup\dots\cup B_k} < \tau^+_{B_{k+1}}\}\big[1+\mathcal{O}(\mathrm{e}^{-\theta/\sigma^2})\big] \in \mathbb{R} && 1 \leqslant k < N \\ |\lambda_k| &< \varrho = 1 - \frac{c}{\log(\sigma^{-1})} && k \geqslant N \end{aligned}$$

    where $\mathring{\pi}_0^{k+1}$ is a certain QSD on B_{k+1} and $c, \theta > 0$
  - kth right eigenfunction ϕ_k close to $\mathbb{P}^x\{\tau_{B_{k+1}} < \tau_{B_1\cup\dots\cup B_k}\}$
  - kth left eigenfunction π_k close to QSD of $K_{(B_1\cup\dots\cup B_k)^c}$

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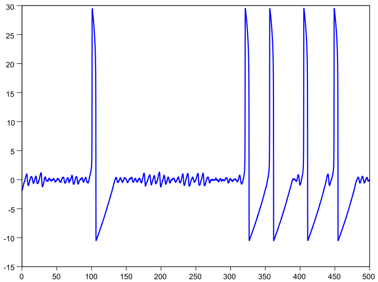
- Expected hitting times:
  $\mathbb{E}^x[\tau_{B_1\cup\dots\cup B_k}] = [1-\lambda_k]^{-1}[1 + \mathcal{O}(\mathrm{e}^{-\kappa/\sigma^2})] \qquad \forall x \in B_{k+1},\ 1 \leqslant k \leqslant N-1$

- Degenerate case: similar to finite chain...

3. FitzHugh–Nagumo equations



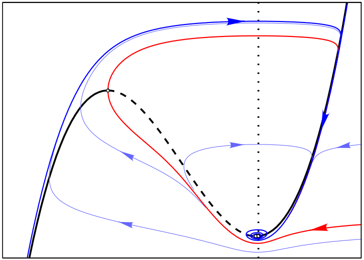
Stochastic FitzHugh–Nagumo equation

$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right]\mathrm{d}t$$ neuron membrane potential

$$\mathrm{d}y_t = \left[a - x_t - by_t\right]\mathrm{d}t$$ open ion channels

▷ $b = 0$ for simplicity in this talk, bifurcation parameter $\delta := \frac{3a^2-1}{2}$

$\varepsilon = 0.1$
$\delta = 0.02$



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$$\mathrm{d}x_t = \frac{1}{\varepsilon}\left[x_t - x_t^3 + y_t\right]\mathrm{d}t + \frac{\sigma_1}{\sqrt{\varepsilon}}\,\mathrm{d}W_t^{(1)}$$ neuron membrane potential

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- $W_t^{(1)}, W_t^{(2)}$: independent Wiener processes
- $0 < \sigma_1, \sigma_2 \ll 1$, $\sigma = \sqrt{\sigma_1^2 + \sigma_2^2}$

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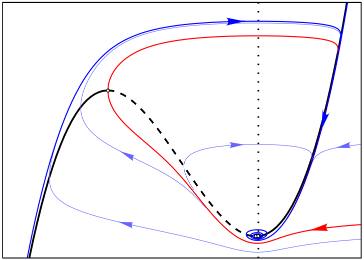

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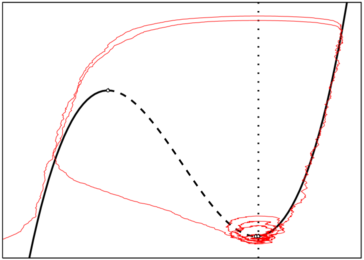
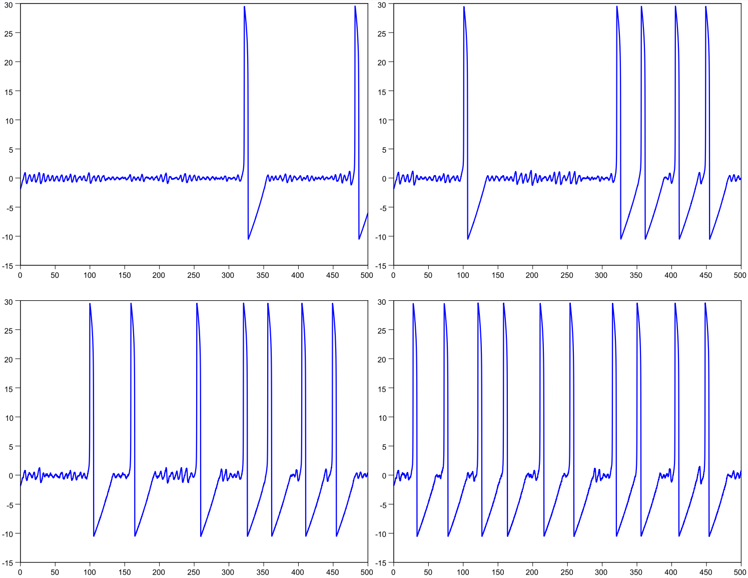
$\varepsilon = 0.1$
$\delta = 0.02$
$\sigma_1 = \sigma_2 = 0.03$

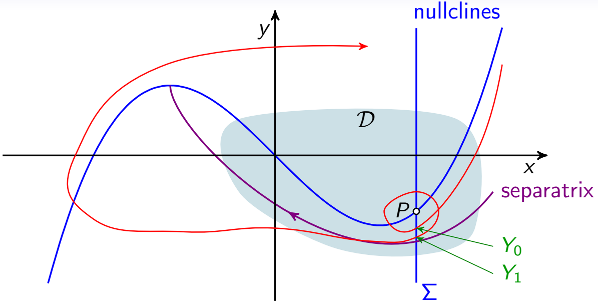


Mixed-mode oscillations (MMOs)



Time series $t \mapsto -x_t$ for $\varepsilon = 0.01$, $\delta = 3 \cdot 10^{-3}$, $\sigma = 1.46 \cdot 10^{-4}, \dots, 3.65 \cdot 10^{-4}$

Random Poincaré map



$Y_0, Y_1, \dots$ substochastic Markov chain describing process killed on $\partial\mathcal{D}$

Number of small oscillations: $N = \inf\{n \geqslant 1 \colon Y_n \notin \Sigma\}$

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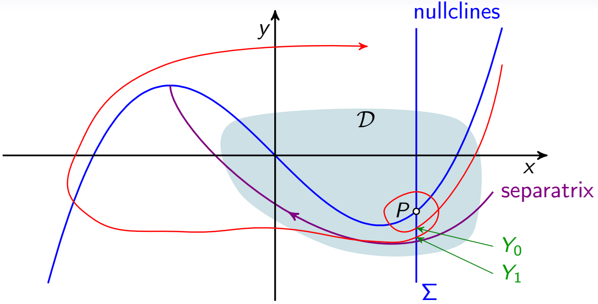

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Theorem 1 [B & Landon, 2012]

N is asymptotically geometric: $\lim_{n\to\infty} \mathbb{P}\{N = n+1 | N > n\} = 1 - \lambda_0$

where $\lambda_0 \in \mathbb{R}_+$: principal eigenvalue of the kernel K, $\lambda_0 < 1$ if $\sigma > 0$

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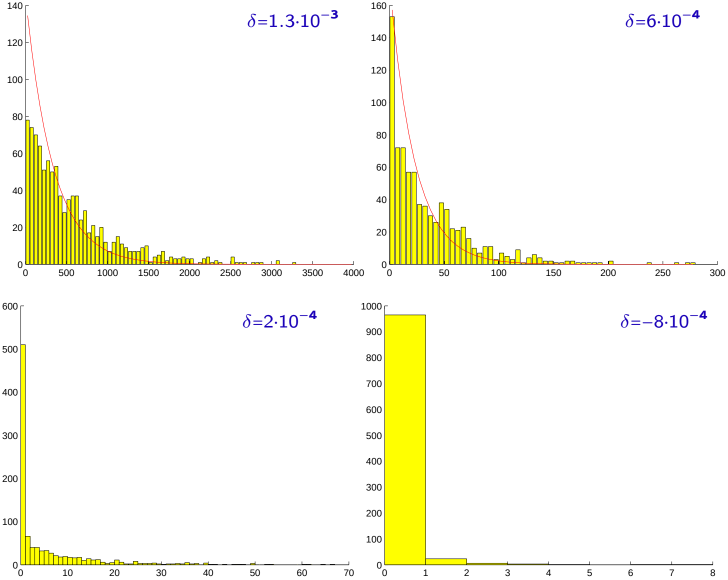
Proof:

Histograms of distribution of N (1000 spikes)

$\sigma = \varepsilon = 10^{-4}$



Weak-noise regime

Theorem 2 [B & Landon, 2012]

Assume ε and $\delta/\sqrt{\varepsilon}$ sufficiently small

There exists $\kappa > 0$ s.t. for $\sigma^2 \leqslant (\varepsilon^{1/4}\delta)^2 / \log(\sqrt{\varepsilon}/\delta)$

- ▷ Principal eigenvalue:

$$1 - \lambda_0 \leqslant \exp\left\{-\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

- ▷ Expected number of small oscillations:

$$\mathbb{E}^{\mu_0}[N] \geqslant C(\mu_0) \exp\left\{\kappa \frac{(\varepsilon^{1/4}\delta)^2}{\sigma^2}\right\}$$

where $C(\mu_0)$ = probability of starting on Σ above separatrix

Proof:

Dynamics near the separatrix

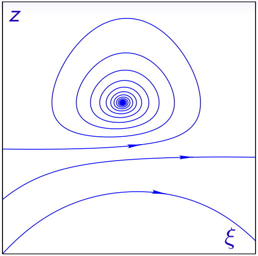
Change of variables:

- ▷ Translate to Hopf bif. point
- ▷ Scale space and time
- ▷ Straighten nullcline $\dot{x} = 0$

$\Rightarrow$ variables (ξ, z) where nullcline: $\{z = \frac{1}{2}\}$

$$\mathrm{d}\xi_t = \Big(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\Big)\,\mathrm{d}t$$

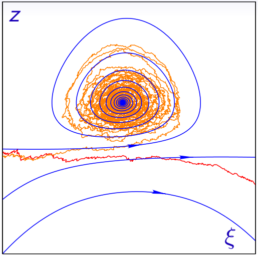
$$\mathrm{d}z_t = \Big(\mu + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\Big)\,\mathrm{d}t$$



where

$$\mu = \frac{\delta}{\sqrt{\varepsilon}}$$

Dynamics near the separatrix



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$$\mathrm{d}\xi_t = \left(\frac{1}{2} - z_t - \frac{\sqrt{\varepsilon}}{3}\xi_t^3\right)\mathrm{d}t + \tilde{\sigma}_1\,\mathrm{d}W_t^{(1)}$$

$$\mathrm{d}z_t = \left(\tilde{\mu} + 2\xi_t z_t + \frac{2\sqrt{\varepsilon}}{3}\xi_t^4\right)\mathrm{d}t - 2\tilde{\sigma}_1\xi_t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

where

$$\tilde{\mu} = \frac{\delta}{\sqrt{\varepsilon}} - \tilde{\sigma}_1^2 \qquad \tilde{\sigma}_1 = -\sqrt{3}\frac{\sigma_1}{\varepsilon^{3/4}} \qquad \tilde{\sigma}_2 = \sqrt{3}\frac{\sigma_2}{\varepsilon^{3/4}}$$

Upward drift dominates if $\tilde{\mu}^2 \gg \tilde{\sigma}_1^2 + \tilde{\sigma}_2^2 \Rightarrow (\varepsilon^{1/4}\delta)^2 \gg \sigma_1^2 + \sigma_2^2$

Rotation around P: use that $2z\,\mathrm{e}^{-2z-2\xi^2+1}$ is constant for $\tilde{\mu} = \varepsilon = 0$

Take $A = \{z > \tilde{\mu}^{1-\gamma}\}$ with $0 < \gamma < \frac{1}{4}$ □

From below to above threshold

Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right)\mathrm{d}t - \tilde{\sigma}_1 t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\biggl(-\pi^{1/4}\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}}\biggr) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{\mathrm{e}^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$

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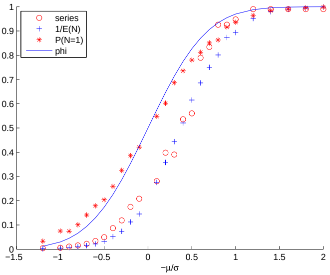
Linear approximation:

$$\mathrm{d}z_t^0 = \left(\tilde{\mu} + t z_t^0\right)\mathrm{d}t - \tilde{\sigma}_1 t\,\mathrm{d}W_t^{(1)} + \tilde{\sigma}_2\,\mathrm{d}W_t^{(2)}$$

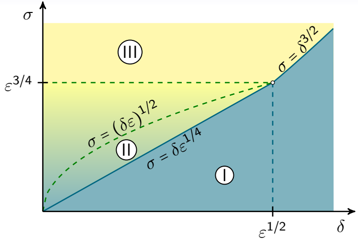
$$\Rightarrow \quad \mathbb{P}\{\text{no small osc}\} \simeq \Phi\Big(-\pi^{1/4}\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}}\Big) \qquad \Phi(x) = \int_{-\infty}^{x} \frac{e^{-y^2/2}}{\sqrt{2\pi}}\,\mathrm{d}y$$



$*$: $\mathbb{P}\{\text{no small osc}\}$
$+$: $1/\mathbb{E}[N]$
$\circ$: $1-\lambda_0$
curve: $x \mapsto \Phi(\pi^{1/4}x)$

$$x = -\frac{\tilde{\mu}}{\sqrt{\tilde{\sigma}_1^2+\tilde{\sigma}_2^2}} = -\frac{\varepsilon^{1/4}(\delta-\sigma_1^2/\varepsilon)}{\sqrt{\sigma_1^2+\sigma_2^2}}$$

Summary: Parameter regimes



$\sigma_1 = \sigma_2$:

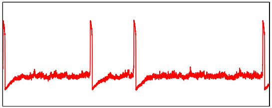
$$\mathbb{P}\{N = 1\} \simeq \Phi\Big(-\frac{(\pi\varepsilon)^{1/4}(\delta-\sigma^2/\varepsilon)}{\sigma}\Big)$$

$$\Phi(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-y^2/2}\,dy$$

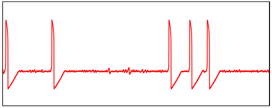
see also

[Muratov & Vanden Eijnden '08]

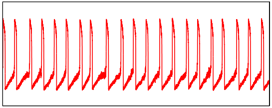
Regime I: rare isolated spikes
Theorem 2 applies ($\delta \ll \varepsilon^{1/2}$)
Interspike interval $\simeq$ exponential



Regime II: clusters of spikes
interspike osc asympt geometric
$\sigma = (\delta\varepsilon)^{1/2}$: geom$(1/2)$



Regime III: repeated spikes
$\mathbb{P}\{N = 1\} \simeq 1$
Interspike interval $\simeq$ constant



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