Lecture on random dynamical systems: a brief introduction

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IRTG Winter School Stochastic Dynamics, Bielefeld

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Organization

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- **Structure of Lectures**
  1. Lecture I: Random Dynamical Systems (RDS): Important definitions and examples
  2. Lecture II: Linear RDS, Multiplicative Ergodic Theorem and Lyapunov exponents
  3. Lecture III: Lyapunov exponents and bifurcations

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(ii) a **model of the dynamics** on X perturbed by noise formalized as a *cocycle* φ over θ, i.e. a *stochastic flow*.

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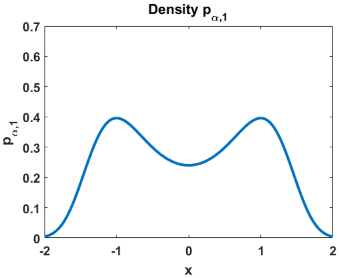
2. **Stochastic Differential Equations** (SDEs)

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sum_j g_j(X_t) \circ \mathrm{d}W_t^j.$$

How is the theory related to classical stochastic analysis?

One-point motion:

Stationary, ergodic density p for SDE



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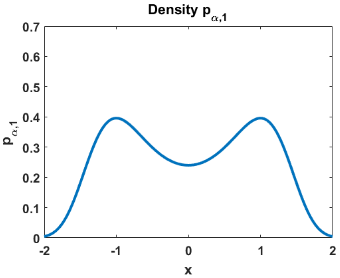
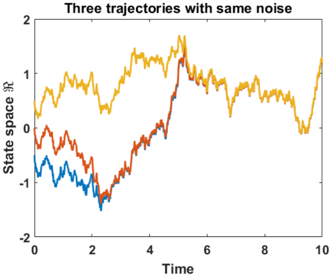
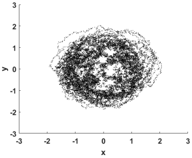


***n*-point motion**:
Random trajectories driven by same noise



Technical core elements of random dynamical systems

1. **Random attractors** with **chaotic** behavior and fractal geometry:



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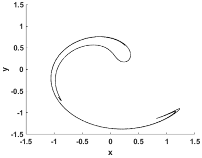
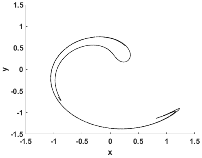


$t = 5$



$t = 50$

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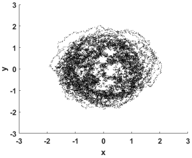

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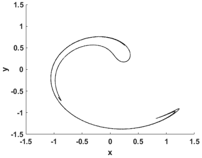

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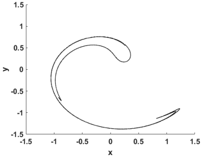

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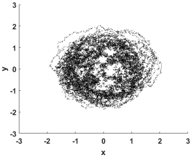

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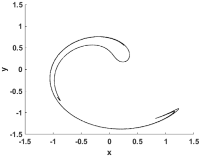

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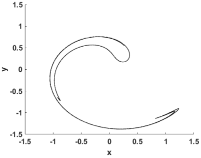

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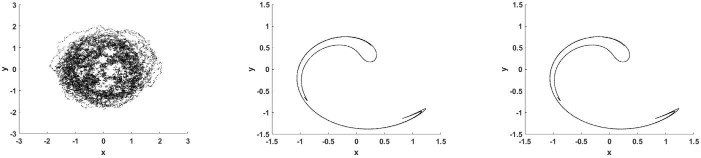

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4. **Stochastic bifurcations**: Toplogical changes of random attractors depending on some system parameter

Main references for Lectures I and II

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