1. Random dynamical systems: Important definitions and examples

1.1 Definition of a random dynamical system

Def.: $(\Omega, \mathcal{F}, \mathbb{P})$ probability space, $(X, \mathcal{B})$ measurable space, $\mathbb{T} \in \{\mathbb{Z}, \mathbb{Z}_0^+, \mathbb{R}, \mathbb{R}_0^+\}$

1. A random dynamical system (RDS) is a pair (Θ, φ) with:

- measurable $\Theta : \mathbb{T} \times \Omega \to \Omega$, $(t, \omega) \mapsto \Theta_t \omega$ s.t.

(measure-preserving DS)

(i) $\Theta_0 = id$, $\Theta_{t+s} = \Theta_t \circ \Theta_s$, $\forall s, t \in \mathbb{T}$

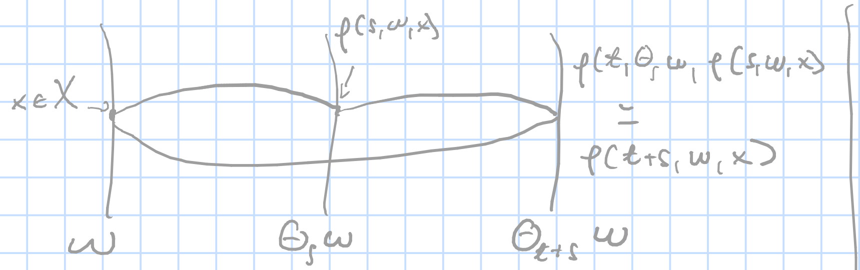
(ii) $\mathbb{P}(A) = \mathbb{P}(\Theta_t^{-1} A)$, $\forall A \in \mathcal{F}, t \in \mathbb{T}$

- meas. $\varphi : \mathbb{T} \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$

(cocycle)

i) $\varphi(0, \omega, \cdot) = id \qquad \forall \omega \in \Omega$

(ii) $\varphi(t+s, \omega, \cdot) = \varphi(t, \Theta_s \omega, \varphi(s, \omega, \cdot))$
$\forall \omega \in \Omega, t, s \in \mathbb{T}$



2. If X topological space with $\mathcal{B} = \mathcal{B}(X)$ (Borel σ-alg.) and

$$\varphi(\cdot, \omega, \cdot) : \mathbb{T} \times X \to X, \quad (t, x) \mapsto \varphi(t, \omega, x)$$

is continuous for all $\omega \in \Omega$:

Then the RDS (θ, φ) is called continuous.

3. If X is a smooth (C^∞) manifold (e.g. $\mathbb{R}^d$)

$$\varphi(t, \omega, \cdot) : X \to X, \quad x \mapsto \varphi(t, \omega, x), \text{ is } C^k:$$

Then the RDS (θ, φ) is called C^k.

1.2 Generating RDS in discrete time

Prop.: (θ, φ) RDS on X, $\mathbb{T} \in \{\mathbb{Z}_0^+, \mathbb{Z}\}$

1. $\mathbb{T} = \mathbb{Z}_0^+$: Consider

$$\boxed{\psi(\omega) := \varphi(1, \omega) : X \to X.}$$

We obtain

$$\varphi(n, \omega) = \begin{cases} \psi(\theta_{n-1}\omega) \circ \dots \circ \psi(\omega), & n \geq 1. \\ id, & n = 0. \end{cases}$$

2. $\mathbb{T} = \mathbb{Z}$: Additionally, we have

$$\boxed{\varphi(-1, \omega) = \varphi(1, \theta_{-1}\omega)^{-1} = \psi(\theta_{-1}\omega)^{-1}}$$

such that $\psi(\omega)$ invertible and

$$\varphi(n, \omega) = \psi(\theta_n \omega)^{-1} \circ \dots \circ \psi(\theta_{-1}\omega)^{-1}, \quad n \leq -1.$$

Proof: Application of cocycle property. □

Continuity/Differentiability of the RDS follow from

— " — of

$x \mapsto \varphi(\omega)x \quad \forall \omega \in \Omega$ (and $x \mapsto \varphi(\omega)^{-1}x$ if $\mathbb{T} = \mathbb{Z}$).

1.2.1 Examples

① Linear RDS as product of random matrices: $X = \mathbb{R}^d$, RDS linear (in x):

$$n \geq 1: \quad \boxed{\varphi(n,\omega) = A_{n-1}(\omega) \cdots A_0(\omega), \quad A_k(\omega) = A(\theta_k \omega)}$$

for some meas. $A : \Omega \to \mathbb{R}^{d \times d}$.

$\to \mathbb{T} = \mathbb{Z}$:

$$n \leq -1: \quad \varphi(n,\omega) = A_n(\omega)^{-1} \cdots A_{-1}(\omega)^{-1}, \quad A_k(\omega) = A(\theta_k \omega).$$

(1*) Herman's example:

$$A(\omega) = \begin{bmatrix} G & 0 \\ 0 & G^{-1} \end{bmatrix} \begin{bmatrix} \cos 2\pi\omega & -\sin 2\pi\omega \\ \sin 2\pi\omega & \cos 2\pi\omega \end{bmatrix},$$

- $\omega \in S^1 \ (\simeq [0,1))$
- $\theta : S^1 \to S^1$ irrational rotation, $(\theta_n := \theta^n)$

keeping Lebesgue meas. invariant

($\to$ later: pos. first Lyapunov exponent)

② Chaos game for Cantor set:

$$x \in X = [0,1]: \quad T_0(x) = \frac{x}{3}, \quad T_1(x) = \frac{2+x}{3}$$

Random switching as an RDS:

- $\Delta = \{0,1\}$, $\Omega = \Delta^{\mathbb{N}} := \{\omega = (\omega_n)_{n=0}^{\infty} \mid \omega_n \in \Delta\}$
- cylinder sets $C_{i_0,\dots,i_n} = \{\omega \in \Omega \mid \omega_\ell = i_\ell, \ell = 0,\dots,n\}$
- Probability measure ν: $\nu(0) = p_0$, $\nu(1) = p_1 = 1 - p_0$
- $(\Omega, \mathcal{B}(\Omega))$ endowed with $\mathbb{P} = \nu^{\mathbb{N}}$

given by $\mathbb{P}(C_{i_0,\dots,i_n}) = p_{i_0} \cdots p_{i_n}$

($\to$ Caratheodory extension)

- Shift map $\theta : \Omega \to \Omega$

$$\theta(\omega_n)_{n=0}^{\infty} = (\omega_{n+1})_{n=0}^{\infty}$$

$\rightarrow$ Cocycle $\varphi : \mathbb{Z}_0^+ \times X \times \Omega \to X$ as

$$\varphi(0,\omega,x) = x, \quad \varphi(n,\omega,x) = T_{i_{n-1}} \circ \dots \circ T_{i_0}(x),$$

where $\omega = (i_\ell)_{\ell=0}^{\infty}$.

Construction can be generalized for random maps drawn from compact, metric space Δ and $\Omega = \Delta^{\mathbb{Z}}$, i.e. two-sided time.

1.2.2 Relation to Markov chains

Theorem: RDS (θ, φ) generated by $\psi(\omega)$

The orbit (x_n^x) given by

$$x_{n+1} = \psi(\theta_n \omega) x_n, \quad x_0 = x \in X,$$

is a Markov chain with

$$P(x, B) = \mathbb{P}(\{\omega : \psi(\omega) x \in B\}) \quad \forall B \in \mathcal{B}.$$

Proof: see notes. $\square$

Remark:

Reverse: Markov chain $\to$ RDS

true under mild conditions (see Kifer '86)

((In general, uniqueness not guaranteed)

1.3 RDS from SDEs

Consider the Stratonovich SDE (midpoint rule)

$$\text{(SDE)} \quad dX_t = f(X_t)dt + \sum_{j=1}^{m} g_j(X_t) \circ dW_t^j, \quad X_0 = x \in \mathbb{R}^d$$

where $f \in C_b^{k,\delta}$, $g_j \in C_b^{k+1,\delta}$, $k \in \mathbb{N}$, $\delta \in (0,1]$,

and W_t^j are independent Brownian motions.

1.3.1 Brownian motion as dynamical system

Def. and Prop.: Wiener space $(\Omega, \mathcal{F}, \mathbb{P})$ is given as follows:

① $\Omega = C_0(\mathbb{R}, \mathbb{R}^d) = \{ \omega : \mathbb{R} \to \mathbb{R}^d \text{ cont.} : \omega(0) = 0 \}$

Ω is endowed with the compact-open topology given by the complete metric

$$\boxed{\begin{gathered}\rho(\omega, \hat{\omega}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega - \hat{\omega}\|_n}{1 + \|\omega - \hat{\omega}\|_n}, \\ \|\omega - \hat{\omega}\| := \sup_{|t| \le n} \|\omega(t) - \hat{\omega}(t)\|.\end{gathered}}$$

(2.) $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel σ-algebra on (Ω, ρ).

(3.) Wiener measure $\mathbb{P}$ on $(\Omega, \mathcal{F})$:

$$\boxed{\mathbb{P}(\{\omega \in \Omega : \omega_1(t) \le x_1, \dots, \omega_d(t) \le x_d\}) = \frac{1}{(2\pi t)^{d/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} e^{-\|y\|^2/2|t|} \, dy_1 \cdots dy_d,}$$

for all $x \in \mathbb{R}^d$.

$\left((W_t^1(\omega), \dots, W_t^d(\omega))^T := \omega(t) \text{ are independent BMs} \right)$

(4.) $\mathcal{F}_s^t \subset \mathcal{F}$ (sub-σ-algebra $\to$ filtration) generated by $\{\omega(u) - \omega(v), \; s \le v \le u \le t\}$.

(5.) Family of shift maps $(\Theta_t)_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$

$$\Theta_t \, \omega(\cdot) = \omega(t + \cdot) - \omega(t)$$

is measure-preserving and ergodic

$(A \in \mathcal{F}, \; \Theta_t^{-1}(A) \subseteq A \implies \mathbb{P}(A) = 0 \text{ or } \mathbb{P}(A) = 1)$

Proof: See notes and Schuss '10. $\square$

1.3.2 SDE solutions as cocycles

Theorem: Consider (SDE) as above: there is a unique meas. map $(t,\omega,x) \mapsto \varphi(t,\omega,x)$ such that

- (θ,φ) is a C^k RDS
- $\varphi(t,\cdot,x)$ solves (SDE).

Proof: See Arnold/Scheutzow '95.

Sketch:

① Establish that solutions of (SDE) give C^k flow of diffeomorphisms $(\varphi_{st}(\omega) := \varphi_{st}(\omega,\cdot))$

$$(2PF) \quad \varphi_{rt}(\omega) \circ \varphi_{sr}(\omega) = \varphi_{st}(\omega), \quad \varphi_{ss}(\omega) = id.$$

(Check: $dX_t = dW_t$ generates C^∞ flow

$x \mapsto \varphi_{st}(\omega,x) = x + W_t(\omega) - W_s(\omega)$) [cf. Kunita '90].

② $$\varphi_{rt}(\omega) = \varphi_{0,t-r}(\theta_r\omega) \quad \text{for almost all } \omega \in \Omega$$

(Check for $dX_t = dW_t$)

$\varphi(t,\cdot) := \varphi_{0t}(\cdot)$, replace t,r,s in (2PF) by $t+s, s, 0$

$\rightarrow \varphi(t+s,\omega) = \varphi(t,\theta_s\omega) \circ \varphi(s,\omega)$ for fixed $s \in \mathbb{R}$, all $t \in \mathbb{R}$ $\mathbb{P}$-a.s.

$\rightarrow$ Crude cocycle: Exceptional null set may depend on s.

③ Perfection of cocycle via abstract argument, using Haar measures. □

1.4 Invariant measures and correspondence theorem

RDS (θ, φ) with time $\mathbb{T}$.

$\rightarrow$ Skew-product flow

$$\bar{\Theta}_t(\omega, x) = (\theta_t \omega, \varphi(t, \omega, x))$$

For measure μ, $T^*\mu(\cdot) = \mu(T^{-1}(\cdot))$ push forward

Def.: Prob. meas. μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ is invariant for (θ, φ) if

(i) $\bar{\Theta}_t^* \mu = \mu$ for all $t \in \mathbb{T}$.

(ii) $$\mu(d\omega, dx) = \mu_\omega(dx)\, \mathbb{P}(d\omega),$$

where $\omega \mapsto \mu_\omega$ is a random/sample measure:

- μ_ω is a prob. meas. on X a.s.
- $\omega \mapsto \mu_\omega(B)$ meas. for all $B \in \mathcal{B}$.

Remark: For X Polish space (complete, separable, metric space):

μ invariant for (θ, φ) iff

$$\varphi(t, \omega, \cdot)^* \mu_\omega = \mu_{\theta_t \omega}, \quad \text{a.s. } \forall t \in \mathbb{T}.$$

Now, consider RDS (θ, φ) with $\mathcal{F}_u^v \subset \mathcal{F}$ generated by $\varphi(t, \theta_s \omega, x)$, for $x \in X$, $t, s \in \mathbb{T}$ with $u \leq s \leq v$ and $0 \leq t \leq v - s$.

$\mathcal{F}_{-\infty}^v$, $\mathcal{F}_u^\infty$ natural extensions.

Def. RDS (Θ, φ) is called Markov if $\mathcal{F}_{-\infty}^{0}$ (the past) and $\mathcal{F}_{0}^{\infty}$ (the future) are independent.

Note: Markov RDS $\Rightarrow$ Markov process;
Markov process and RDS (e.g. (SDE)) $\Rightarrow$ Markov RDS

Def.: Invariant meas. μ for (Θ, φ) is Markov if $\{\mu_\omega\}$ is $\mathcal{F}_{-\infty}^{0}$-meas.

Theorem [Correspondence of measures]

(Θ, φ) Markov RDS on Polish space X:

(a) If μ invar. Markov meas. with samples μ_ω, then $\rho = \mathbb{E}[\mu_\omega]$ stationary for the Markov process ($P_t^* \rho = \rho$ for dual semi-group)

$\oplus$: μ' s.t. $\mathbb{E}[\mu_\omega] = \mathbb{E}[\mu'_\omega] \Rightarrow \mu = \mu'$

(b) ρ stationary: Then for $t_n \to \infty$, almost surely

$$\mu_\omega = \lim_{n \to \infty} \varphi(t_n, \Theta_{-t_n}\omega, \cdot)^* (\rho)$$

weakly.

$\to$ μ invariant Markov meas. and $\rho = \mathbb{E}[\mu_\omega]$.

Summarizing: one-to-one correspondence between stationary and invariant Markov meas.

Proof: See Lecture Notes (or directly Kuksin/Shirikyan '12). $\square$

Note: $\mathbb{E}[\mu_\omega(\cdot) \mid \mathcal{F}_0^\infty] = \mathbb{E}[\mu_\omega(\cdot)] = \rho(\cdot)$

$$\Rightarrow \mathbb{E}[\mu(\cdot) \mid \mathcal{F}_0^\infty] = (\mathbb{P} \times \rho)(\cdot).$$

Example: O-U process:

$$(OU) \quad dX_t = -\gamma X_t \, dt + \sigma \, dW_t, \quad \gamma > 0.$$

Stationary $\rho(dx) = \underbrace{C}_{\text{normalization}} e^{-\gamma x^2/\sigma^2} \, dx$

$\rightarrow$ gives invariant Markov meas. μ for Markov RDS induced by (OU).