Freie Universität Berlin, Department of Mathematics

Lecture Notes on Random Dynamical Systems

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Main references

The main references for this lecture are [2, 59, 61], but there will be a list of several other references that are updated throughout the course.

Chapter 0

Some elements of measure theory and dynamical systems

0.1 Measures and measure spaces

0.1.1 Basic definitions and properties

We collect the most basic definitions in measure theory, followed by some results which will be useful in the lectures.

Definition 0.1.1 (Algebra and σ-algebra)**.** Consider a collection $\mathcal{A}$ of subsets of a set X with $\emptyset \in \mathcal{A}$, and the following properties:

(a) When $A \in \mathcal{A}$ then $A^c := X \setminus A \in \mathcal{A}$.

(b) When $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

(b') Given a finite or infinite sequence $\{A_k\}$ of subsets of X, $A_k \in \mathcal{A}$, then also $\bigcup_k A_k \in \mathcal{A}$.

If $\mathcal{A}$ satisfies (a) and (b), it is called an *algebra* of subsets of X; if it satisfies (a) and (b'), it is called a *σ-algebra* .

It follows from the definition that a σ-algebra is an algebra, and for an algebra $\mathcal{A}$ holds

- $\emptyset, X \in \mathcal{A}$;
- $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$;
- $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$;
- if $\mathcal{A}$ is a σ-algebra , then $\{A_k\} \subset \mathcal{A} \Rightarrow \bigcap_k A_k \in \mathcal{A}$.

Definition 0.1.2 (Measure)**.** A function $\mu : \mathcal{A} \to [0, \infty]$ on a σ-algebra $\mathcal{A}$ is a *measure* if

(a) $\mu(\emptyset) = 0$;

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$; and

(c) $\mu\left(\bigcup_k A_k\right) = \sum_k \mu(A_k)$ if $\{A_k\}$ is a finite or infinite sequence of pairwise disjoint sets from $\mathcal{A}$, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$. This property of μ is called *σ-additivity* (or *countable additivity*).

If, in addition, $\mu(X) = 1$, then μ is called a *probability measure*.

Definition 0.1.3.

(a) If $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a *measure space*. The subsets of X contained in $\mathcal{A}$ are called *measurable*.

(b) If $\mu(X) < \infty$ (resp. $\mu(X) = 1$) then the measure space is called *finite* (resp. *probabilistic* or *normalized*).

(c) If there is a sequence $\{A_k\} \subset \mathcal{A}$ satisfying $X = \bigcup_k A_k$ and $\mu(A_k) < \infty$ for all k, then the measure space $(X, \mathcal{A}, \mu)$ is called *σ-finite*.

A set $N \in \mathcal{A}$ with $\mu(N) = 0$ is called a *null set*. If a certain property involving the points of a measure space holds true except for a null set, we say the property holds *almost everywhere* (we write a.e., which, depending on the context, sometimes means "almost every"). We also use the word *essential* to indicate that a property holds a.e. (e.g., "essential bijection").

Definition 0.1.4. The σ-algebra generated by a collection $\mathcal{A}_0$ of subsets of X, also denoted by $\sigma(\mathcal{A}_0)$, is the smallest σ-algebra containing $\mathcal{A}_0$, i.e.

$$\sigma(\mathcal{A}_0) = \bigcap_{\mathcal{A} \text{ is a } \sigma\text{-algebra with } \mathcal{A}_0 \subseteq \mathcal{A}} \mathcal{A}.$$

Given two measurable spaces $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$, the σ-algebra generated by the products of subsets of X_1 and X_2, i.e.,

$$A_1 \otimes A_2 := \sigma(\{A_1 \times A_2 \,:\, A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\})$$

is called the product σ-algebra .

Analogously we can define the algebra of subsets of X generated by some collection of subsets of X.

Theorem 0.1.5 (Hahn–Kolmogorov extension theorem)**.** *Let X be a set, $\mathcal{A}_0$ an algebra of subsets of X, and $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ a σ-additive function. If $\mathcal{A}$ is the σ-algebra generated by $\mathcal{A}_0$, there exists a measure $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu\big|_{\mathcal{A}_0} = \mu_0$. If μ_0 is σ-finite, the extension is unique.*

This result becomes especially useful if we would like to define measures on sets of sequences.

Definition 0.1.6 (Cylinder)**.** Let $\mathcal{A}_k$ be a σ-algebra for $k \in \mathrm{N}$. Let $k_1 < k_2 < \ldots < k_r$ be integers and $A_{k_i} \in \mathcal{A}_{k_i}$, $i = 1, \ldots, r$. A *cylinder set* (also called *rectangle*) is a set of the form

$$[A_{k_1}, \ldots, A_{k_r}] = \{\{x_j\}_{j \in \mathrm{N}} \,:\, x_{k_i} \in A_{k_i},\ 1 \leq i \leq r\}\,.$$

Definition 0.1.7. Let $(X_i, \mathcal{A}_i, \mu_i)$, $i \in \mathrm{N}$, be normalized measure spaces. The *product measure space* $(X, \mathcal{A}, \mu) = \prod_{i\in\mathrm{N}}(X_i, \mathcal{A}_i, \mu_i)$ is defined by

$$X = \prod_{i\in\mathrm{N}} X_i \quad \text{and} \quad \mu\left([A_{k_1}, \ldots, A_{k_r}]\right) = \prod_{j=1}^{r} \mu_{k_j}(A_{k_j}).$$

An analogous definition holds if we replace N by $\mathbb{Z}$, i.e., if X consists of bi-infinite sequences.

One can see that finite unions of cylinders form an algebra of subsets of X. By Theorem 0.1.5 it can be uniquely extended to a measure on $\mathcal{A}$, the smallest σ-algebra containing all cylinders.

It is often necessary to approximate measurable sets by sets of some sub-class (e.g., an algebra) of the given σ-algebra :

Theorem 0.1.8. *Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $\mathcal{A}_0$ be an algebra of subsets of X generating $\mathcal{A}$. Then, for each $\varepsilon > 0$ and each $A \in \mathcal{A}$ there is some $A_0 \in \mathcal{A}_0$ such that $\mu(A \triangle A_0) < \varepsilon$. Here, $E \triangle F := (E \setminus F) \cup (F \setminus E)$ denotes the* symmetric difference *of E and F.*

0.1.2 The monotone class theorem

Definition 0.1.9. As sequence of sets $\{A_k\}$ is called *increasing* (resp. *decreasing*) if $A_k \subseteq A_{k+1}$ (resp. $A_k \supseteq A_{k+1}$) for all k.
The notation $A_k \uparrow A$ (resp. $A_k \downarrow A$) means that $\{A_k\}$ is an increasing (resp. decreasing) sequence of sets with $\bigcup_k A_k = A$ (resp. $\bigcap_k A_k = A$).

Definition 0.1.10 (Monotone class)**.** Let X be a set. A collection $\mathcal{M}$ of subsets of X is a *monotone class* if whenever $A_k \in \mathcal{M}$ and $A_k \uparrow A$, then $A \in \mathcal{M}$.

Theorem 0.1.11 (Monotone Class Theorem)**.** *A monotone class which contains an algebra, also contains the σ-algebra generated by this algebra.*

Thus, if we show that sets with a certain property form a monotone class, and this class contains an algebra $\mathcal{A}$ of sets, then it contains $\sigma(\mathcal{A})$. For instance, if we can show that two measures μ, ν coincide on an algebra, they coincide on the whole σ-algebra generated by it. This holds true because $\{\mu = \nu\}$ is a monotone class.

Chapter 1

Random dynamical systems and their generators

1.1 Basic definitions

Firstly, we define what we mean by a random dynamical system throughout this lecture. We will consider systems in discrete and continuous time, one-and two-sided. Hence, in the following we will always assume that the index set $\mathbb{T}$ satisfies

$$\mathbb{T} \in \left\{\mathbb{R}, \mathbb{R}_0^+, \mathbb{Z}, \mathbb{Z}_0^+\right\}.$$

A random dynamical system on a measurable space $(X, \mathcal{B})$ consists of

(i) a model of the noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, formalised as a measurable flow $(\theta_t)_{t\in\mathbb{T}}$ of $\mathbb{P}$-preserving transformations $\theta_t : \Omega \to \Omega$,

(ii) a model of the dynamics on X perturbed by noise formalised as a *cocycle* φ over θ.

In technical detail, the definition of a random dynamical system is given as follows:

Definition 1.1.1 (Random dynamical system)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X, \mathcal{B})$ be a measurable space.

1. A random dynamical system (RDS) is a pair of mappings (θ, φ) such that the following holds:

   - The $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F},\ \mathcal{F})$-measurable mapping $\theta : \mathbb{T} \times \Omega \to \Omega$, $(t, \omega) \mapsto \theta_t\omega$, is a *metric dynamical system*, i.e.
     (i) $\theta_0 = \text{id}$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for $t, s \in \mathbb{T}$,
     (ii) $\mathbb{P}(A) = \mathbb{P}(\theta_t^{-1}A)$ for all $A \in \mathcal{F}$ and $t \in \mathbb{T}$.
   - The $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F} \otimes \mathcal{B},\ \mathcal{B})$-measurable mapping $\varphi : \mathbb{T} \times \Omega \times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$, is a cocycle over θ, i.e., $\varphi(0, \omega, \cdot) = \text{id}$ and

$$\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot)) \quad \text{for all } \omega \in \Omega \text{ and } t, s \in \mathbb{T}. \tag{1.1.1}$$

2. If X is a topological space with $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra, and

$$(t, x) \mapsto \varphi(t, \omega, x)$$

is continuous for every $\omega \in \Omega$, the random dynamical system (θ, φ) is called *continuous*.

3. If X is additionally a smooth, i.e. C^∞, d-dimensional manifold (e.g. $\mathbb{R}^d$), and for each $(t, \omega) \in \mathbb{T} \times \Omega$ the mapping

$$\varphi(t, \omega) := \varphi(t, \omega, \cdot) : X \to X, \; x \mapsto \varphi(t, \omega, x)$$

is C^k, i.e. k-times differentiable in x and the derivatives are continuous in (t, x), the random dynamical system (θ, φ) is called C^k.

We still speak of a random dynamical system, if its cocycle is only defined in forward time, i.e., if the mapping φ is only defined on $\mathbb{R}_0^+ \times \Omega \times X$ or $\mathbb{Z}_0^+ \times \Omega \times X$, while the underlying metric dynamical system is defined in forward and backward time, i.e., the mappings θ_t are defined for all $t \in \mathbb{R}$ or $t \in \mathbb{Z}$ respectively. We will make it noticeable whenever this is the case.

Remark 1.1.2. In the following, the metric dynamical system $(\theta_t)_{t \in \mathbb{T}}$ is often even ergodic, i.e. any $A \in \mathcal{F}$ with $\theta_t^{-1} A = A$ for all $t \in \mathbb{T}$ satisfies $\mathbb{P}(A) \in \{0, 1\}$.

Remark 1.1.3. Further, note that the trajectories of the RDS might explode in finite time. In this case one can consider it as a *local* random dynamical system (as opposed to the *global* random dynamical system from Definition 1.1.1) being defined only for times bounded by some random explosion times $\tau^-(\omega, x)$ and $\tau^+(\omega, x)$. We will consider local RDS in more detail in the context of Chapter 5.

We state our first theorem on two-sided random dynamical systems.

Theorem 1.1.4. *Consider an RDS (θ, φ) on a measurable space $(X, \mathcal{B})$ and two-sided time set $\mathbb{T}$, i.e., $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.*

(a) For all $(t, \omega) \in \mathbb{T} \times \Omega$, the cocycle $\varphi(t, \omega)$ is a bimeasurable bijection of $(X, \mathcal{B})$ and,

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega) \;\; \text{for all} \;\; (t, \omega) \in \mathbb{T} \times \Omega,$$

or, equivalently,

$$\varphi(-t, \omega) = \varphi(t, \theta_{-t} \omega)^{-1} \;\; \text{for all} \;\; (t, \omega) \in \mathbb{T} \times \Omega,$$

Furthermore, the mapping

$$(t, \omega, x) \mapsto \varphi(t, \omega)^{-1} x$$

is measurable.

1. *If X is a topological space and the RDS is continuous, then for all $(t, \omega) \in \mathbb{T} \times \Omega$ we have that $\varphi(t, \omega) : X \to X$ is a homeomorphism. If*

(a) $\mathbb{T} = \mathbb{Z}$, *or*

(b) $\mathbb{T} = \mathbb{R}$, *and* X *is a compact Hausdorff space,*

then additionally $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ *is continuous for all* $\omega \in \Omega$.

2. *If* X *is a smooth manifold and the RDS is* C^k, *then for all* $(t, \omega) \in \mathbb{T} \times \Omega$ *we have that* $\varphi(t, \omega) : X \to X$ *is a diffeomorphism. Moreover,* $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ *is* C^k *with respect to* x *for all* $\omega \in \Omega$.

Proof. See Exercise sheet 1. □

Before we address the question of how such random dynamical systems are generated, we introduce a distinction that will be highly relevant when we discuss random dynamical systems in the context of stochastic differential equations. Recall the cocycle property (1.1.1), which in this form is called the *perfect* cocycle property. If equation (1.1.1) holds for fixed $s \in \mathbb{T}$ and all $t \in \mathbb{T}$, $\mathbb{P}$-a.s., where the expectional set N_s with $\mathbb{P}(N_s) = 0$ may depend on s, we call φ a *crude* cocycle. If equation (1.1.1) holds for fixed $s, t \in \mathbb{T}$, $\mathbb{P}$-a.s., where the expectional set $N_{s,t}$ with $\mathbb{P}(N_{s,t}) = 0$ may depend on s, t, we call φ a *very crude* cocycle. The perfection of a very crude cocycle is easy to observe in discrete time but will require some work in continuous time:

Theorem 1.1.5 (Perfection for discrete time)**.** *Let* φ *be a very crude cocycle over* θ *with discrete time* $\mathbb{T}$. *Then there exists a cocycle* ψ *over* θ *which is perfect and indistinguishable from* φ, *i.e., there exists a set* $N \in \mathcal{F}$ *with* $\mathbb{P}(N) = 0$ *and*

$$\{\omega : \psi(t, \omega) \neq \varphi(t, \omega) \text{ for some } t \in \mathbb{T}\} \subset N.$$

Proof. See Exercise sheet 1. □

1.2 Random dynamical systems from products of random mappings

In this section, we focus on random dynamical systems in discrete time $\mathbb{T} \in \{\mathbb{Z}, \mathbb{Z}_0^+\}$. Since, typically, the family of measure-preserving transformations $(\theta_n)_{n \in \mathbb{T}}$ consists of iterations of a map $\theta : \Omega \to \Omega$, we adopt the notation $(\theta^n)_{n \in \mathbb{T}}$ for this section.

Firstly, we make the following observation:

Proposition 1.2.1. *Let* (θ, φ) *be an RDS on* X *with time* $\mathbb{T} \in \{\mathbb{Z}_0^+, \mathbb{Z}\}$.

1. *If* $\mathbb{T} = \mathbb{Z}_0^+$, *we introduce the* time-one mapping

$$\psi(\omega) := \varphi(1, \omega) : X \to X, \tag{1.2.1}$$

and obtain

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \mathrm{id}, & n = 0. \end{cases} \tag{1.2.2}$$

The RDS is measurable if and only if $(\omega, x) \mapsto \psi(\omega)x$ *is measurable. It is continuous/*C^k *if and only if* $x \mapsto \psi(\omega)x$ *is continuous/*C^k*. Conversely, given a family of mappings* $\psi(\omega) : X \to X$ *such that* $(\omega, x) \mapsto \psi(\omega)x$ *is measurable/continuous/*C^k*, then* φ *defined by* (1.2.2) *is the cocycle of a measurable/continuous/*C^k *RDS. We say that* φ *is generated by* ψ*.*

2. *If* $\mathbb{T} = \mathbb{Z}$*, we additionally have the* time-minus-one mapping

$$\varphi(-1, \omega) = \varphi(1, \theta^{-1}\omega)^{-1} = \psi(\theta^{-1}\omega)^{-1} \tag{1.2.3}$$

such that $\psi(\omega) : X \to X$ *is invertible and we obtain*

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \mathrm{id}, & n = 0, \\ \psi(\theta^n\omega)^{-1} \circ \cdots \circ \psi(\theta^{-1}\omega)^{-1}, & n \leq -1, \end{cases} \tag{1.2.4}$$

The RDS is measurable if and only if

$$(\omega, x) \mapsto \psi(\omega)x \quad \text{and} \quad (\omega, x) \mapsto \psi(\omega)^{-1}x \tag{1.2.5}$$

are measurable. It is continuous/C^k *if and only if* $x \mapsto \psi(\omega)x$ *is a homeomorphism/diffeomorphism of order* k*. Conversely, given a family of invertible mappings* $\psi(\omega) : X \to X$ *such that the mappings* (1.2.5) *are measurable/continuous/*C^k*, then* φ *defined by* (1.2.4) *is the cocycle of a measurable/continuous/*C^k *RDS.*

Proof. Straight-forward application of the cocycle property (1.1.1). □

We can put on record: every one-sided (two-sided) discrete time RDS has the form (1.2.2) ((1.2.4)), also called *product of random mappings* or *iterated function system.* Note that we can write the discrete time cocycle $\varphi(n, \omega, x)$ as the solutions of an initial value problem for a random difference equation

$$x_{n+1} = \psi(\theta^n\omega)x_n, \; n \in \mathbb{T}, \; x_0 = x \in X. \tag{1.2.6}$$

Consider the following examples:

Example 1.2.2. *1.* Linear random dynamical system as product of random matrices*: If* $X = \mathbb{R}^d$ *and the RDS is linear, we can write for* $n \geq 1$

$$\varphi(n, \omega) = A_{n-1}(\omega) \cdots A_0(\omega), \; A_k(\omega) = A(\theta^k\omega),$$

where $A : \Omega \to \mathbb{R}^{d \times d}$ *is measurable. Two-sided linear RDS correspond with invertible measurable families of matrices, giving in addition for* $n \leq -1$

$$\varphi(n, \omega) = A_n(\omega)^{-1} \cdots A_{-1}(\omega)^{-1}, \; A_k(\omega) = A(\theta^k\omega).$$

2. Barnsley's chaos game*: Note that one can approximate a Cantor set by randomly switching between the maps*

$$T_0(x) = \frac{x}{2}, \qquad T_1(x) = \frac{1+x}{2}$$

on $X = [0,1]$. Such a switching between random maps can be formalized as an RDS by considering the finite set $\Delta = \{0,1\}$ and the (topological) space of sequences

$$\Omega \equiv \Delta^{\mathbb{N}} := \{\omega = (\omega_n)_{n=0}^{\infty} \mid \omega_n \in \Delta\}.$$

Recall that a cylinder set is of the form

$$C_{i_0,i_1,\dots,i_n} = \{\omega \in \Omega \mid \omega_k = i_k,\ k = 0,1,\dots,n\},$$

for some $n \in \mathbb{N}$. Having the two probabilities $1 > p_1 = 1 - p_0 > 0$, we endow the measurable space $(\Omega, \mathcal{B}(\Omega))$ with the infinite product measure $\mathbb{P}$, defined uniquely by its action on cylinder sets as

$$\mathbb{P}(C_{i_0,i_1,\dots,i_n}) = p_{i_0} \cdots p_{i_n}.$$

The metric dynamical system is given by iterations of the shift map $\theta : \Omega \to \Omega$ defined as

$$\theta(\omega_n)_{n=0}^{\infty} = (\omega_{n+1})_{n=0}^{\infty}.$$

The evolution of the system through time is given by applying the map T_0 or T_1 with probabilities p_0 or p_1, respectively, and this is expressed by the cocycle $\varphi : \mathbb{Z}_0^+ \times X \times \Omega \to X$ as

$$\varphi(0,\omega,x) = x, \qquad \varphi(n,\omega,x) = T_{i_{n-1}} \circ \cdots T_{i_0}(x),$$

where $\omega = (i_k)_{k=0}^{\infty}$. (See also Exercise sheet 1.)

For discrete-time random dynamical systems with independent increments, we can prove the following relation to Markov chains:

Theorem 1.2.3. *Let φ be a measurable cocycle over θ with time $\mathbb{T} = \mathbb{Z}_0^+$, generated by $\psi(\omega)$ such that the sequence $\psi(\theta^n \cdot)$ is identically and independently distributed. Then, given any random variable x_0, the orbit (x_n^x) given by*

$$x_{n+1} = \psi(\theta^n \omega) x_n,\ x_0 = x \in X,$$

is a time-homogeneous Markov chain on X with transition probability

$$P(x,B) = \mathbb{P}\{\omega : \psi(\omega)x \in B\} \quad \text{for all } B \in \mathcal{B}. \tag{1.2.7}$$

Proof. Firstly, note that $P(x,B)$ as defined in (1.2.7) is, indeed, a Markov kernel: $\mathbb{P}(x,\cdot)$ is a probability measure on $(X,\mathcal{B})$ by definition. Furthermore, we observe that $\mathbb{P}(\cdot,B)$ is a measurable map for any $B \in \mathcal{B}$, as follows: Introducing $\Psi : (\omega,x) \mapsto \psi(\omega)x$ and writing

$A_x = \{\omega \in \Omega : (\omega, x) \in A\}$ for $A \in \mathcal{F} \otimes \mathcal{B}$, we have

$$P(x, B) = \mathbb{P}(A_x), \; A = \Psi^{-1} B \in \mathcal{F} \otimes \mathcal{B}.$$

We observe by the monotone class theorem that

$$\mathcal{A} = \{A \in \mathcal{F} \otimes \mathcal{B} : \mathbb{P}(A_x) \text{ is measurable in } x\}$$

is a σ-algebra and, hence, $\mathcal{A} = \mathcal{F} \otimes \mathcal{B}$.

Let us denote $\mathcal{F}_n = \sigma(x_0^x, \dots, x_n^x; x \in X)$. Since $1_B(\psi(\theta^n \cdot)x)$ is independent from $\mathcal{F}_n$ for each $x \in X$ and $B \in \mathcal{B}$, we can deduce by the well-know properties of conditional expectations that

$$\mathbb{P}(x_{n+1}^x \in B | \mathcal{F}_n) = \mathbb{E}[1_B(\psi(\theta^n \omega) x_n^x) | \mathcal{F}_n] = \mathbb{E}[1_B(\psi(\theta^n \omega) x_n^x) | x_n^x] = \mathbb{P}(x_{n+1}^x \in B | x_n^x).$$

This shows the Markov property. Moreover, we obtain the time-homogeneity

$$\mathbb{P}(x_{n+1}^x \in B | x_n^x = y) = \mathbb{P}(\omega : \psi(\theta^n \omega) y \in B) = \mathbb{P}(\omega : \psi(\omega) y \in B) = P(y, B),$$

having used the θ^n-invariance of $\mathbb{P}$ for all $n \geq 1$. □

Remark 1.2.4. The reverse direction, i.e., the consruction of a discrete-time random dynamical system as a composition of independent random maps from a Markov chain with given transition probabilities, is also possible (see [59, Theorem 1.1]), but, in general, uniqueness cannot be guaranteed. This has to do with the RDS perspective of providing a description of the n-point motion, i.e., tracking trajectories with different initial conditions but driven by the same noise, whereas the Markov chain only describes the 1-point motion. We will discuss this distinction in more detail later on.

[**End of Lecture I, 13.04.**]

1.3 Random dynamical systems from random differential equations

1.4 Random dynamical systems from stochastic differential equations

Throughout this thesis, we will investigate random dynamical systems induced by stochastic differential equations. Hence, we are interested in random dynamical systems adapted to a suitable filtration and of white noise type. Following [47], we make the following definition:

Definition 1.4.1 (White noise RDS)**.** Let (θ, φ) be a random dynamical system over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a topological space X where φ is defined in forward time. Let

$(\mathcal{F}_s^t)_{-\infty\leq s\leq t\leq\infty}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that

(i) $\mathcal{F}_t^u \subset \mathcal{F}_s^v$ for all $s \leq t \leq u \leq v$,

(ii) $\mathcal{F}_s^t$ is independent from $\mathcal{F}_u^v$ for all $s \leq t \leq u \leq v$,

(iii) $\theta_r^{-1}(\mathcal{F}_s^t) = \mathcal{F}_{s+r}^{t+r}$ for all $s \leq t$, $r \in \mathbb{R}$,

(iv) $\varphi(t,\cdot,x)$ is $\mathcal{F}_0^t$-measurable for all $t \geq 0$ and $x \in X$.

Furthermore we denote by $\mathcal{F}_{-\infty}^t$ the smallest *sigma*-algebra containing all $\mathcal{F}_s^t$, $s \leq t$, and by $\mathcal{F}_t^\infty$ the smallest *sigma*-algebra containing all $\mathcal{F}_t^u$, $t \leq u$. Then (θ,φ) is called a *white noise (filtered) random dynamical system.*

Consider a stochastic differential equation (SDE)

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t)dW_t,\ X_0 \in \mathbb{R}^d\,, \tag{1.4.1}$$

where (W_t) denotes some r-dimensional standard Brownian motion, the drift $f : \mathbb{R}^d \to \mathbb{R}^d$ is a locally Lipschitz continuous vector field and the diffusion coefficient $g : \mathbb{R}^d \to \mathbb{R}^{d\times r}$ a Lipschitz continuous matrix-valued map. If in addition f satisfies a bounded growth condition, as for example a one-sided Lipschitz condition, then by [37] there is a white noise random dynamical system (θ,φ) associated to the diffusion process solving (1.4.1). The probabilistic setting is as follows: We set $\Omega = C_0(\mathbb{R},\mathbb{R}^r)$, i.e. the space of all continuous functions $\omega : \mathbb{R} \to \mathbb{R}^r$ satisfying that $\omega(0) = 0 \in \mathbb{R}^r$. If we endow Ω with the compact open topology given by the complete metric

$$\kappa(\omega,\widehat{\omega}) := \sum_{n=1}^\infty \frac{1}{2^n}\frac{\|\omega-\widehat{\omega}\|_n}{1+\|\omega-\widehat{\omega}\|_n}, \quad \|\omega-\widehat{\omega}\|_n := \sup_{|t|\leq n}\|\omega(t)-\widehat{\omega}(t)\|\,,$$

we can set $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel-sigma algebra on (Ω,κ). There exists a probability measure $\mathbb{P}$ on $(\Omega,\mathcal{F})$ called *Wiener measure* such that the r processes $(W_t^1),\ldots,(W_t^r)$ defined by $(W_t^1(\omega),\ldots,W_t^r(\omega))^{\mathrm{T}} := \omega(t)$ for $\omega \in \Omega$ are independent one-dimensional Brownian motions. Furthermore, we define the sub-σ-algebra $\mathcal{F}_s^t$ as the σ-algebra generated by $\omega(u)-\omega(v)$ for $s \leq v \leq u \leq t$. The ergodic metric dynamical system $(\theta_t)_{t\in\mathbb{R}}$ on $(\Omega,\mathcal{F},\mathbb{P})$ is given by the shift maps

$$\theta_t : \Omega \to \Omega, \quad (\theta_t\omega)(s) = \omega(s+t)-\omega(t)\,.$$

Indeed, these maps form an ergodic flow preserving the probability $\mathbb{P}$, see e.g. [2].

In chapter **??**, we are not able to work with a one-sided Lipschitz condition. Instead, we will use a transformation into a random differential equation to show that the respective stochastic differential equation induces a random dynamical system.

Multidimensional conversion formula from Stratonovich to Itô integral

Consider the Stratonovich SDE

$$\mathrm{d}X_t = \overline{F}(t,X_t)\mathrm{d}t + G(t,X_t)\circ\mathrm{d}W_t\,,$$

where $\overline{F} : \mathbb{R}^d \to \mathbb{R}^d$ is called the drift of the SDE, $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ the diffusion of the SDE and W_t is an m-dimensional Wiener process. In accordance with [45], the equation has the same solutions as the Itô SDE

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}W_t\,,$$

where

$$\overline{F}_i(t, X_t) = F_i(t, X_t) - \frac{1}{2}\sum_{j=1}^{d}\sum_{k=1}^{m} G_{jk}(t, X_t)\frac{\partial G_{ik}}{\partial X_j}(t, X_t)\,, \quad i = 1, \dots, d\,.$$

The Fokker–Planck equation

Consider the Itô SDE

$$\mathrm{d}X_t = F(t, X_t)\mathrm{d}t + G(t, X_t)\mathrm{d}W_t\,,$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ is called the drift of the SDE and $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ the diffusion of the SDE and W_t is an m-dimensional Wiener process. The so called Fokker–Planck equation describes the evolution of the density of the process $(X_t)_{t\geq 0}$ under sufficient (classical or Sobolev) regularity of the coefficients:

$$\frac{\partial p(t, x)}{\partial t} = -\sum_{i=1}^{d}\frac{\partial}{\partial x_i}[F_i(t, x)p(x, t)] + \frac{1}{2}\sum_{i,j=1}^{d}\frac{\partial^2}{\partial x_i \partial x_j}[D_{ij}(x, t)p(x, t)]\,,$$

with diffusion tensor

$$D_{ij}(x, t) = \sum_{k=1}^{m} G_{ik}(x, t)G_{jk}(x, t)\,.$$

1.5 Invariant measures

Let (θ, φ) be a random dynamical system with the cocycle φ being defined on one-or two-sided time $\mathbb{T} \in \{\mathbb{R}_0^+, \mathbb{R}\}$. Then the system generates a skew product flow, i.e. a family of maps $(\Theta_t)_{t\in\mathbb{T}}$ from $\Omega \times X$ to itself such that for all $t \in \mathbb{T}$ and $\omega \in \Omega, x \in X$

$$\Theta_t(\omega, x) = (\theta_t\omega, \varphi(t, \omega, x))\,.$$

The notion of an invariant measure for the random dynamical system is given via the invariance with respect to the skew product flow, see e.g. [2, Definition 1.4.1]. We denote by $T\mu$ the push forward of a measure μ by a map T, i.e. $T\mu(\cdot) = \mu(T^{-1}(\cdot))$.

Definition 1.5.1 (Invariant measure)**.** A probability measure μ on $\Omega \times X$ is invariant for the random dynamical system (θ, φ) if

(i) $\Theta_t\mu = \mu$ for all $t \in \mathbb{T}$,

(ii) the marginal of μ on Ω is $\mathbb{P}$, i.e. μ can be factorised uniquely into $\mu(\mathrm{d}\omega, \mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$ where $\omega \mapsto \mu_\omega$ is a *random measure* (or *disintegration* or *sample measure*) on X, i.e. μ_ω

is a probability measure on X for $\mathbb{P}$-a.a. $\omega \in \Omega$ and $\omega \mapsto \mu_\omega(B)$ is measurable for all $B \in \mathcal{B}(X)$.

The marginal of μ on the probability space is demanded to be $\mathbb{P}$ since we assume the model of the noise to be fixed. Note that the invariance of μ is equivalent to the invariance of the random measure $\omega \mapsto \mu_\omega$ on the state space X in the sense that

$$\varphi(t,\omega,\cdot)\mu_\omega = \mu_{\theta_t\omega} \quad \mathbb{P}\text{-a.s. for all } t \in \mathbb{T}\,. \tag{1.5.1}$$

For white noise random dynamical systems (θ,φ), in particular random dynamical systems induced by a stochastic differential equation, there is a one-to-one correspondence between certain invariant random measures and stationary measures of the associated stochastic process, first observed in [30]. In more detail, we can define a Markov semigroup $(P_t)_{t\geq 0}$ by setting

$$P_t f(x) = \mathbb{E}(f(\varphi(t,\cdot,x))$$

for all measurable and bounded functions $f : X \to \mathbb{R}$. If $\omega \mapsto \mu_\omega$ is a $\mathcal{F}_{-\infty}^0$-measurable invariant random measure in the sense of (1.5.1), also called *Markov measure*, then

$$\rho(\cdot) = \mathbb{E}[\mu_\omega(\cdot)] = \int_\Omega \mu_\omega(\cdot)\mathbb{P}(d\omega)$$

turns out to be an invariant measure for the Markov semigroup $(P_t)_{t\geq 0}$, often also called stationary measure for the associated process. If ρ is an invariant measure for the Markov semigroup, then

$$\mu_\omega = \lim_{t\to\infty} \varphi(t,\theta_{-t}\omega,\cdot)\rho$$

exists $\mathbb{P}$-a.s. and is an $\mathcal{F}_{-\infty}^0$-measurable invariant random measure.

We observe similarly to [10] that, in the situation of μ and ρ corresponding in the way described above,

$$\mathbb{E}[\mu_\omega(\cdot)|\mathcal{F}_0^\infty] = \mathbb{E}[\mu_\omega(\cdot)] = \rho(\cdot)\,,$$

and, hence,

$$\mathbb{E}[\mu(\cdot)|\mathcal{F}_0^\infty] = (\mathbb{P} \times \rho)(\cdot)\,.$$

Therefore the probability measure $\mathbb{P} \times \rho$ is invariant for $(\Theta_t)_{t\geq 0}$ on $(\Omega \times X, \mathcal{F}_0^\infty \times \mathcal{B}(X))$. In words, the product measure with marginals $\mathbb{P}$ and ρ is invariant for the random dynamical system restricted to one-sided path space. We will discuss a similar relation for quasi-stationary and quasi-ergodic measures in Chapter 5.

Chapter 2

Spectral theories of random dynamical systems

2.1 Lyapunov exponents and the Multiplicative Ergodic Theorem

Fundamental for stochastic bifurcation theory is Oseledets' Multiplicative Ergodic Theorem, which implies the existence of Lyapunov exponents describing stability properties of a differentiable random dynamical system.

The random dynamical system (θ, φ) is called C^k if $\varphi(t, \omega, \cdot) \in C^k$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$, where again $\mathbb{T} \in \{\mathbb{R}, \mathbb{R}_0^+\}$. Let's assume that X is a smooth d-dimensional manifold and that (θ, φ) is C^1. The *linearisation* or *derivative* $\mathrm{D}\varphi(t, \omega, x)$ of $\varphi(t, \omega, \cdot)$ at $x \in X$ is a linear map from the tangent space T_x to the tangent space $T_{\varphi(t,\omega,x)}$. If $X = \mathbb{R}^d$, the linearisation is simply the Jacobian $d \times d$ matrix

$$\mathrm{D}\varphi(t, \omega, x) = \frac{\partial \varphi(t, \omega, x)}{\partial x}\,.$$

Differentiating the equation

$$\varphi(t + s, \omega, x) = \varphi(t, \theta_s \omega, \varphi(s, \omega, x))$$

on both sides and applying the chain rule to the right hand side yields

$$\mathrm{D}\varphi(t + s, \omega, x) = \mathrm{D}\varphi(t, \theta_s \omega, \varphi(s, \omega, x))\mathrm{D}\varphi(s, \omega, x) = \mathrm{D}\varphi(t, \Theta_s(\omega, x))\mathrm{D}\varphi(s, \omega, x)\,,$$

i.e. the cocycle property of $\mathrm{D}\varphi$ with respect to the skew product flow $(\Theta_t)_{t \in \mathbb{T}}$.

Let us now assume that the random dynamical system possesses an invariant measure μ. In case $X = \mathbb{R}^d$, this implies that $(\Theta, \mathrm{D}\varphi)$ is a (potentially one-sided) random dynamical system with linear cocycle $\mathrm{D}\varphi$ over the metric dynamical system $(\Omega \times X, \mathcal{F} \otimes \mathcal{B}(X), (\Theta_t)_{t \in \mathbb{T}})$, see e.g. [2, Proposition 4.2.1]. Generally, we have that $\mathrm{D}\varphi$ is a linear bundle random dynamical system on the tangent bundle TX (see [2, Definition 1.9.3, Proposition 4.25]).

In large parts of this work, we will be concerned with a stochastic differential equation in Stratonovich form

$$\mathrm{d}X_t = f_0(X_t)\mathrm{d}t + \sum_{i=1}^{m} f_j(X_t) \circ \mathrm{d}W_t^j \tag{2.1.1}$$

where W_t^j are independent real valued Brownian motions, f_0 is a C^1 vector field and $f_1, \dots, f_m$ are C^2 vector fields satisfying bounded growth conditions, as e.g. (global) Lipschitz continuity, in all derivatives to guarantee the existence of a (global) random dynamical system for φ and $\mathrm{D}\varphi$. We write the equation in Stratonovich form when differentiation is concerned as the classical rules of calculus are preserved. If $X = \mathbb{R}^d$, we can apply the conversion formula to the Itô integral (see Appendix 1.4) to obtain the situation of (1.4.1).

According to [6], the derivative $\mathrm{D}\varphi(t,\omega,x)$ applied to an initial condition $v_0 \in T_xX \cong \mathbb{R}^d$ solves uniquely the variational equation

$$\mathrm{d}v = \mathrm{D}f_0(\varphi(t,\omega,x))v\,\mathrm{d}t + \sum_{j=1}^{m} \mathrm{D}f_j(\varphi(t,\omega,x))v \circ \mathrm{d}W_t^j\,, \quad v(0) = v_0 \in T_xX\,. \tag{2.1.2}$$

In case the derivative can be written as a matrix, as for example for $X = \mathbb{R}^d$, the Jacobian $\mathrm{D}\varphi(t,\omega,x)$ satisfies Liouville's equation

$$\begin{aligned} \det \mathrm{D}\varphi(t,\omega,x) = \exp\Bigg(&\int_0^t \operatorname{trace} \mathrm{D}f_0(\varphi(s,\omega,x))\mathrm{d}s \\ &+ \sum_{j=1}^{m} \int_0^t \operatorname{trace} \mathrm{D}f_j(\varphi(s,\omega,x)) \circ \mathrm{d}W_s^j \Bigg). \end{aligned} \tag{2.1.3}$$

We summarise the different versions of the Multiplicative Ergodic Theorem for differentiable random dynamical systems in one-sided and two-sided time in the following theorem [2, Theorem 3.4.1, Theorem 3.4.11, Theorem 4.2.6], containing all the properties we will need in the following.

Theorem 2.1.1 (Multiplicative Ergodic Theorem)**.** *(a) Suppose the C^1-random dynamical system (θ, φ), where φ is defined in forward time, has an ergodic invariant measure ν and satisfies the integrability condition*

$$\sup_{0 \le t \le 1} \ln^+ \|\mathrm{D}\varphi(t,\omega,x)\| \in L^1(\nu).$$

Then there exist a Θ-invariant set $\Delta \subset \Omega \times X$ with $\nu(\Delta) = 1$, a number $1 \le p \le d$ and real numbers $\lambda_1 > \cdots > \lambda_p$, the Lyapunov exponents with respect to ν, such that for all $0 \neq v \in T_xX \cong \mathbb{R}^d$ and $(\omega, x) \in \Delta$

$$\lambda(\omega,x,v) := \lim_{t\to\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t,\omega,x)v\| \in \{\lambda_p, \dots, \lambda_1\}\,.$$

Furthermore, the tangent space $T_xX \cong \mathbb{R}^d$ admits a filtration

$$\mathbb{R}^d = V_1(\omega,x) \supsetneq V_2(\omega,x) \supsetneq \cdots \supsetneq V_p(\omega,x) \supsetneq V_{p+1}(\omega,x) = \{0\}\,,$$

for all $(\omega, x) \in \Delta$ *such that*

$$\lambda(\omega, x, v) = \lambda_i \quad \Longleftrightarrow \quad v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \dots, p\}.$$

In case the derivative can be written as a matrix, we have for all $(\omega, x) \in \Delta$

$$\lim_{t \to \infty} \frac{1}{t} \ln \det \mathrm{D}\varphi(t, \omega, x) = \sum_{i=1}^{p} d_i \lambda_i, \tag{2.1.4}$$

where d_i *is the multiplicity of the Lyapunov exponent* λ_i *and* $\sum_{i=1}^{p} d_i = d$.

(b) *If the cocycle* φ *is defined in two-sided time and satisfies the above integrability condition also in backwards time, there exists the Oseledets splitting*

$$\mathbb{R}^d = E_1(\omega, x) \oplus \cdots \oplus E_p(\omega, x)$$

of the tangent space into random subspaces $E_i(\omega, x)$*, the Oseledets spaces, for all* $(\omega, x) \in \Delta$*. These have the following properties for all* $(\omega, x) \in \Delta$*:*

(i) *The Oseledets spaces are invariant under the derivative flow, i.e. for all* $t \in \mathbb{R}$

$$\mathrm{D}\varphi(t, \omega, x) E_i(\omega, x) = E_i(\Theta_t(\omega, x)),$$

(ii)

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, x) v\| = \lambda_i \iff v \in E_i(\omega, x) \setminus \{0\} \quad \text{for all } i \in \{1, \dots, p\},$$

(iii) *The dimension equals the multiplicity of the associated Lyapunov exponent, i.e.*

$$\dim E_i(\omega, x) = d_i.$$

2.2 The Furstenberg–Khasminskii formula

The standard method for deriving an explicit formula of the largest Lyapunov exponent λ_1 is given by the *Furstenberg-Khasminskii formula* which will play a crucial role in Chapters **??** and 5, and also partly in Chapter **??**. We give a short introduction based on [52]. For more detailed discussions we refer to [2, 4].

Consider the linear Stratonovich equation

$$\mathrm{d}Y_t = A_0 Y_t \,\mathrm{d}t + \sum_{j=1}^{m} A_j Y_t \circ \mathrm{d}W_t^j, \quad Y_0 = v \in \mathbb{R}^d, \tag{2.2.1}$$

where $A_0, \dots, A_m \in \mathbb{R}^{d \times d}$ and $W^1, \dots, W^m$ are independent Wiener processes in two-sided time. For keeping things simple, we let $A_0, \dots, A_m \in \mathbb{R}^{d \times d}$ not depend on an underlying random system. Thus, equation (2.2.1) induces a linear cocycle Φ over the family of shifts $(\theta_t)_{t\in\mathbb{R}}$ on the Wiener space Ω, as opposed to the previous section where the probability space was $\Omega \times X$ for some manifold X and the skew product flow $(\Theta_t)_{t\in\mathbb{T}}$ replaced $(\theta_t)_{t\in\mathbb{R}}$. We will see in the course of this work that the calculations below are still applicable in such situations.

We introduce the change of variables $r_t = \|Y_t\|$ and $s_t = Y_t/r_t$, so that s_t lies on the unit sphere $\mathbb{S}^{d-1}$. The stochastic differential equation in polar coordinates is given by

$$\mathrm{d}s_t = (A_0 s_t - \langle s_t, A_0 s_t\rangle s_t)\,\mathrm{d}t + \sum_{j=1}^{m}(A_j s_t - \langle s_t, A_j s_t\rangle s_t) \circ \mathrm{d}W_t^j \,,$$

and

$$\mathrm{d}r_t = \langle s_t, A_0 s_t\rangle r_t\,\mathrm{d}t + \sum_{j=1}^{m}\langle s_t, A_j s_t\rangle r_t \circ \mathrm{d}W_t^j \,,$$

Since Stratonovich integration obeys the classical chain rule we can write

$$r_t = r_0 \exp\left(\int_0^t \langle s_\tau, A_0 s_\tau\rangle \mathrm{d}\tau + \sum_{j=1}^{m}\int_0^t \langle s_\tau, A_j s_\tau\rangle \circ \mathrm{d}W_\tau^j\right) .$$

Using the Itô-Stratonovich conversion we obtain

$$r_t = r_0 \exp\left(\int_0^t \left[h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau)\right] \mathrm{d}\tau + \sum_{j=1}^{m}\int_0^t \langle s_\tau, A_j s_\tau\rangle\,\mathrm{d}W_\tau^j\right) , \tag{2.2.2}$$

where

$$\begin{aligned} h_A(s) &= \langle s, As\rangle \,, \\ k_A(s) &= \frac{1}{2}\langle (A+A^*)s, As\rangle - \langle s, As\rangle^2 \,. \end{aligned}$$

It is well known that the Itô integrals in (2.2.2) are of order $(\sqrt{t})$ for large t. Hence, we can conclude that

$$\frac{1}{t}\ln r_t = \frac{1}{t}\int_0^t \left[h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau)\right] \mathrm{d}\tau + \mathcal{O}(t^{-1/2}) \,. \tag{2.2.3}$$

We define

$$g_A(s) = As - \langle s, As\rangle s \quad \text{for } A \in \mathbb{R}^{d\times d}, s \in \mathbb{S}^{d-1} \,,$$

and denote by $\mathfrak{L}(g_{A_0}, \dots, g_{A_m})(s)$ the Lie algebra generated by these vector fields at s. We impose the classical Hörmander condition on the hypoellpticity of these vector fields driving the dynamics of s_t:

$$\dim \mathfrak{L}(g_{A_0}, \dots, g_{A_m})(s) = d-1 \quad \text{for all } s \in \mathbb{S}^{d-1} \,. \tag{2.2.4}$$

Note that the objects in Theorem 2.1.1 only depend on $\omega \in \Omega$ in our situation. According to [51], condition 2.2.4 guarantees that the distribution of the Oseledets space $E_i(\omega)$ in Theorem 2.1.1

(b) possesses a smooth density for any $i \in \{1, \dots, p\}$. Therefore any initial point $v \in \mathbb{R}^d \setminus \{0\}$ has almost surely a non-vanishing component in $E_1(\omega)$ (and $v \in V_1(\omega) \setminus V_2(\omega)$ almost surely in the situation of Theorem 2.1.1 (a)), and, hence,

$$\lambda_1 = \lim_{t\to\infty} \frac{1}{t} \ln r_t \quad \text{almost surely.}$$

The hypoellipticity condition (2.2.4) further implies irreducibility of the Markov semigroup induced by $(s_t)_{t\geq 0}$ on $\mathbb{S}^{d-1}$, and since the unit sphere is a compact manifold, we can conclude that $(s_t)_{t\geq 0}$ possesses a unique stationary probability measure with smooth density p. The density p solves the stationary Fokker-Planck equation

$$\mathcal{L}^* p = 0\,,$$

where

$$\mathcal{L} = g_{A_0} + \frac{1}{2} \sum_{j=1}^{m} g_{A_j}^2$$

is the generator of $(s_t)_{t\geq 0}$ in Hörmander notation and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. By Birkhoff's Ergodic Theorem we observe that

$$\begin{aligned}\lim_{t\to\infty} \frac{1}{t} \ln r_t &= \lim_{t\to\infty} \frac{1}{t} \int_0^t \left[h_{A_0}(s_\tau) + \sum_{j=1}^{m} k_{A_j}(s_\tau) \right] \mathrm{d}\tau \\ &= \int_{\mathbb{S}^{d-1}} \left[h_{A_0}(s) + \sum_{j=1}^{m} k_{A_j}(s) \right] p(s)\, \mathrm{d}s\,.\end{aligned}$$

The Furstenberg–Khasminskii formula for the top Lyapunov exponent is therefore given by

$$\lambda_1 = \int_{\mathbb{S}^{d-1}} \left[h_{A_0}(s) + \sum_{j=1}^{m} k_{A_j}(s) \right] p(s)\, \mathrm{d}s\,. \tag{2.2.5}$$

2.3 Stable and unstable manifolds

Chapter 3

Random attractors

3.1 Basic definitions

Let (θ, φ) be a white noise random dynamical system on a metric space (X, d). We give the definition of a random attractor of (θ, φ) with respect to tempered sets.

A random variable $R : \Omega \to \mathbb{R}$ is called *tempered* if

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \ln^+ R(\theta_t \omega) = 0 \quad \text{for almost all } \omega \in \Omega \,,$$

see also [2, p. 164]. A set $D \in \mathcal{F} \otimes \mathcal{B}(X)$ is called *tempered* if there exists a tempered random variable R such that

$$D(\omega) \subset B_{R(\omega)}(0) \quad \text{for almost all } \omega \in \Omega \,,$$

where $D(\omega) := \{x \in X : (\omega, x) \in D\}$. D is called compact if $D(\omega) \subset X$ is compact for almost all $\omega \in \Omega$. Denote by $\mathcal{D}$ the set of all compact tempered sets $D \in \mathcal{F} \otimes \mathcal{B}(X)$. We now define the notion of a random attractor with respect to $\mathcal{D}$, see also [60, Definition 14.3].

Definition 3.1.1 (Random attractor)**.** A set $A \in \mathcal{D}$ is called a *random attractor* (with respect to $\mathcal{D}$) if the following two properties are satisfied.

(i) A is φ-invariant, i.e.

$$\varphi(t, \omega) A(\omega) = A(\theta_t \omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega \,.$$

(ii) For all $D \in \mathcal{D}$, we have

$$\lim_{t \to \infty} \operatorname{dist}\big(\varphi(t, \theta_{-t}\omega) D(\theta_{-t}\omega), A(\omega)\big) = 0 \quad \text{for almost all } \omega \in \Omega \,,$$

where $\operatorname{dist}(E, F) := \sup_{x \in E} \inf_{y \in F} d(x, y)$.

The set A is called a *weak random attractor* if it satisfies the latter property with almost sure convergence replaced by convergence in probability. We call A a *(weak) random point attractor* if it satisfies the properties above with tempered random sets D replaced by single points $y \in X$

in (ii). A (weak) random point attractor is said to be *minimal* if it is contained in each (weak) random point attractor.

Remark 3.1.2. Note that we require that the random attractor is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$, in contrast to a weaker statement normally used in the literature (see also [33, Remark 4]).

Remark 3.1.3. Property (ii) is sometimes demanded only for compact subsets $B \subset X$ as for example in [47]. Note that any random attractor according to our definition is a random attractor according to this weaker definition.

The existence of random attractors is proved via so-called absorbing sets. A set $B \in \mathcal{D}$ is called an *absorbing set* if for almost all $\omega \in \Omega$ and any $D \in \mathcal{D}$, there exists a $T > 0$ such that

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset B(\omega) \quad \text{for all } \; t \geq T\,.$$

A proof of the following theorem can be found in [48, Theorem 3.5].

Theorem 3.1.4 (Existence of random attractors)**.** *Suppose that (θ, φ) is a continuous random dynamical system with an absorbing set B. Then there exists a unique random attractor A, given by*

$$A(\omega) := \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)} \quad \textit{for almost all } \; \omega \in \Omega.$$

Furthermore, $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}_{-\infty}^{0}$, i.e. the past of the system.

Remark 3.1.5. Naturally, random attractors are related to invariant probability measures of a random dynamical system (θ, φ). It follows directly from [31, Proposition 4.5] that, if the fibres of a random attractor A, i.e. $\omega \mapsto A(\omega)$, are measurable with respect to $\mathcal{F}_{-\infty}^{0}$, there is an invariant measure μ for (θ, φ) such that $\omega \mapsto \mu_\omega$ is measurable with respect to $\mathcal{F}_{-\infty}^{0}$, i.e. is a Markov measure, and satisfies $\mu_\omega(A(\omega)) = 1$ for almost all $\omega \in \Omega$. In particular, if there exists a unique invariant probability measure ρ for the Markov semi-group $(P_t)_{t \geq 0}$, then the invariant Markov measure, supported on A, is unique by the one-to-one correspondence explained above. Additionally, if the Markov semi-group is strongly mixing, i.e.

$$P_t f(x) \xrightarrow{t \to \infty} \int_X f(y)\rho(\mathrm{d}y) \quad \text{for all } \; \text{continuous and bounded } f : X \to \mathbb{R} \text{ and } x \in X\,,$$

then the set $\tilde{A} \in \mathcal{F} \times \mathbb{B}(X)$, given by $\tilde{A}(\omega) = \operatorname{supp} \mu_\omega \subset A(\omega)$ for almost all $\omega \in \Omega$, is a minimal weak random point attractor according to [47, Proposition 2.20].

3.2 Ergodic theory of chaotic random attractors

This chapter is dedicated to shedding more light on the character of the random (point) attractor A and the invariant random measures μ_ω supported on the fibres $A(\omega)$, in case the top Lyapunov exponent λ_1 is positive. We have called A a random strange attractor in this situation and refer to Figures ?? and ?? for getting an idea of the shape of such an attractor. In Figures ??–?? we approximate the support of μ_ω for model (??) with high shear intensity b by computing $\varphi(T, \theta_{-T}\omega)\tilde{\rho}$ for increasing T, where $\tilde{\rho}$ is a numerical approximation of the stationary measure ρ. In Figure ?? we do exactly the same for model (??). In both cases, we observe one-dimensional structures that resemble Henon-like attractors.

Hence, a first thing to show is that μ_ω is almost surely not supported on a singleton if $\lambda_1 > 0$; a fact we have already claimed in Corollary ??. We will follow work by Baxendale and Stroock [14, 10] extending their results to the non-compact case. This will be the content of Section 3.2.1 where we show Theorem 3.2.1 which states that positive Lyapunov exponents imply atomless invariant measures, in particular in the situations of Chapters ?? and ??.

Section 3.2.2 concerns Pesin's formula, i.e. the equality of metric entropy and the sum of positive Lyapunov exponents, for model (??) on $\mathbb{R}^d$ and model (??) on $\mathbb{R} \times \mathbb{S}^1$. The formula was proven by Ledrappier & Young [69] for discrete-time random dynamical systems generated by randomly drawn diffeomorphisms on a compact manifold with absolutely continuous stationary measure. Biskamp [15] has proven the formula for discrete-time random dynamical systems $\mathcal{X}^+(\mathbb{R}^d, \nu)$, where ν is the law of the random diffeomorphisms on $\mathbb{R}^d$, given a class of Assumptions (A1)-(A5) and an absolutely continuous stationary (not necessarily ergodic) probability measure μ. The formula reads

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \mu(\mathrm{d}x)\,,$$

where h_μ denotes metric entropy and $\lambda_i(x)^+$ are the positive Lyapunov exponents with multiplicities $m_i(x)$. The section gives an introduction to the concept of entropy for random dynamical systems, following [15, 59, 73], and links discrete-time systems with random dynamical systems generated by stochastic differential equations via their time-one maps. This allows to formulate Theorem 3.2.10 as a direct consequence of [15, Theorem 9.1] stating Pesin's formula for systems derived from stochastic differential equations. Finally, we prove Corollary 3.2.11 which says that the random dynamical system induced by model (??) has positive entropy for large enough shear or noise respectively, i.e. if $\lambda_1 > 0$. The same would hold true for model (??), once positive Lyapunov exponents can be established.

Section 3.2.5 introduces the concept of SRB measures for random systems with positive Lyapunov exponent. In short, the random measures μ_ω are called SRB measures if they are absolutely continuous with respect to the Riemannian measure on fibres of unstable manifolds. Hence, if the μ_ω can be shown to be SRB measures, their supports are non-singular subsets of the closures of unstable manifolds, perhaps even equal them. This gives a strong characterisation of the shapes we observe in Figures ?? and ??. Following [69], we formulate Theorem 3.2.17 and Corollary 3.2.18 implying the SRB property for sample measures of discrete-time systems and systems induced by stochastic differential equations on compact manifolds, in case the stationary measure μ is absolutely continuous. It is beyond the scope of this work to show the results for the non-compact case since there are a lot of technical intricacies to be taken care of. We simply refer to the intuition that if the maps/flows and their derivatives satisfy uniform bounds, the results are applicable to the non-compact case.

We concede that, too a large extent, the work presented in this chapter is not original. However, it is important to embed the results of chapters **??** and **??** into the context of ergodic theory. Furthermore, we understand this chapter as a contribution to linking the Arnold school, on which the other chapters are based, to the school represented by Kifer, Ledrappier and Young that puts more emphasis on smooth ergodic theory.

3.2.1 Positive Lyapunov exponents imply atomless invariant measures

We are extending the statement in [10] about positive Lyapunov exponents implying atomless invariant random measures to the non-compact setting. Let M be a connected smooth Riemannian manifold of dimension N. Similarly to (2.1.1), we consider the Stratonovich stochastic differential equation on M

$$d\xi_t(x) = X_0(\xi_t(x))\mathrm{d}t + \sum_{k=1}^{d} X_k(\xi_t(x)) \circ \mathrm{d}W_t^k \,, \quad \xi_0(x) = x \,. \tag{3.2.1}$$

Here $X_0, X_1, \dots, X_d$ are smooth vector fields on M satisfying a one-sided Lipschitz condition guaranteeing unique solutions and $\{W_t^k : t \geq 0\}, 1 \leq k \leq d$, are independent real valued Brownian motions on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let further denote $\tilde{X}_k$ the natural lifts of the vector fields X_k on M to vector fields on SM, the unit sphere bundle in TM. For $v \in SM$ and $u \in C([0,\infty), \mathbb{R}^d)$, let $\Psi(\cdot, v, u)$ denote the curve in SM satisfying

$$\dot{\Psi}(t,v,u) = \tilde{X}_0(\Psi(t,v,u)) + \sum_{k=1}^{d} u_k(t)\tilde{X}_k(\Psi(t,v,u))$$

with $\Psi(0,v,u) = v$. Now we demand the following assumption similarly to (2.2.4) which implies that the generator of (3.2.1) is hypoelliptic and that, if there is a stationary probability measure ρ, it is unique and has a smooth density with respect to the Riemannian measure on M:

(A1) $\left(\tilde{X}_0, \dots, \tilde{X}_d\right)(v) = T_v SM$
and $\{\Psi(t,v,u) : t \geq 0 \text{ and } u \in C([0,\infty), \mathbb{R}^d)\}$ is dense in SM for all $v \in SM$.

Equation (3.2.1) induces a random dynamical system as before where the notations identify as $\varphi(t,\cdot,x) = \xi_t(x)$. Let μ be the invariant Markov measure of the RDS, i.e. the invariant probability measure of the skew product flow corresponding with ρ, and μ_ω its disintegrations to the state space. We denote $\bar{\mu} = \mathbb{E}[\mu_\omega \times \mu_\omega]$ which is a stationary probability measure for the two-point motion $\{(\xi_t(x), \xi_t(y)) : t \geq 0\}$ on $M \times M$. The generator of the two-point motion is denoted by $L^{(2)}$. Further, Δ denominates the diagonal in $M \times M$ and we write $\hat{M} = M \times M \setminus \Delta$.

Before we can establish the statement about atomless measures, we have to introduce the moment Lyapunov function $\tilde{\Lambda} : \mathbb{R} \to \mathbb{R}$. It is defined by

$$\tilde{\Lambda}(p) = \lim_{t\to\infty} \frac{1}{t} \ln \mathbb{E}_\mu \left|D\xi_t(x)(v)\right|^p \,, \; v \in T_x M \,, |v| \neq 0 \,.$$

If M is compact, Assumption (A1) guarantees that the above limit exists and is independent from the choice of v. Further $\tilde{\Lambda}$ is a convex analytic function of p according to results in [14]. These results are directly applicable to the situation of model (**??**) since the variational equation is solely dependent on the angular process which is defined on the compact manifold $\mathbb{S}^1$.

We formulate the result in the following theorem. Note that the proof contains a summary of arguments coming from [14, 10, 58] that cannot be found in such an overview otherwise. The issues with non-compactness are rather benign and are pointed out at the respective parts of the argument:

Theorem 3.2.1 (Positive λ_1 implies atomless μ_ω)**.** *Assume that the top Lyapunov exponent λ_1 for the random dynamical system induced by* (3.2.1) *is positive, that the moment Lyapunov function $\tilde{\Lambda}$ exists independently from $v \in TM$, that the system has a stationary measure ρ and that (A1) holds. Then*

$$\bar{\mu}\left(\hat{M}\right) = 1$$

or equivalently μ_ω is atomless almost surely.

Proof. Define the map $\Phi : TM \to M \times M$ by

$$\Phi((x, v)) = (x, \exp_x(v)) \,,$$

where $\exp_x : T_xM \to M$ is the exponential map. For the strategy of the proof it is essential that there exists an injectivity radius $\delta_0 > 0$ such that for any $0 < \delta \leq \delta_0$, Φ is a diffeomorphism from $\{(x, v) \in TM \,:\, 0 < |v| < \delta\}$ onto $\hat{M}_\delta := \{(x, y) \in M^2 \,:\, 0 < d(x, y) < \delta\}$ via

$$(x, \theta, r) \in SM \times (0, \delta) \mapsto \Phi((x, r\theta)) \in \hat{M}_\delta \,.$$

In the original setting of [14] this follows immediately from compactness. For non-compact manifolds with positive injectivity radius, as for example the infinite cylinder or $\mathbb{R}^N$, the result carries over immediately, as we will see from the following arguments. In case of no positive injectivity radius being guaranteed, fix $\varepsilon > 0$. In order to adapt the proof to this setting, we start with a compact ball K such that $(\rho \times \rho)(K \times K) > 1 - \varepsilon/3$. Writing $\hat{K} = K \times K \setminus \Delta$, we proceed as in the proof of [10, Remark 4.12].

Observe that $\tilde{\Lambda}(0) = 0$ and $\tilde{\Lambda}'(0) = \lambda_1 > 0$. From [14], we further know that $\tilde{\Lambda}(-N) \geq 0$ and that $\tilde{\Lambda}$ is convex. Hence, there is a $p < 0$ such that $\tilde{\Lambda}(p) < 0$. Denote the injectivity radius of K by δ_0. Now, according to [14, Theorem 3.18] there is a $\delta \leq \delta_0$ and $\phi_p \in C^\infty(SM \times (0, \delta))$ such that

$$L^{(2)}\phi_p \leq \tilde{\Lambda}(p)\phi_p \text{ and } Cr^p \leq \phi_p(x, \theta, r) \,, \quad (x, \theta, r) \in SM \times (0, \delta) \,, \tag{3.2.2}$$

where $C > 0$ is some constant. By the coordinate transformation from above, we take $V : \hat{K} \to \mathbb{R}$ to be a smooth non-negative extension of ϕ_p with smooth compactly supported extension $\tilde{V}$ to $\hat{M}$. Since $p, \tilde{\Lambda}(p) < 0$ we obtain from (3.2.2) that V satisfies

$$L^{(2)}V(x, y) \to -\infty \text{ as } d(x, y) \to 0 \,,$$

and

$$\int_{\hat{M}} \tilde{V} \mathrm{d}(\rho \times \rho) < \infty.$$

This allows to apply Khasminskii's pointwise estimate in the proof of [58, Theorem 3.7] to the distance of the two-point motion from the diagonal for any $(x, y) \in \hat{K}$. As the upper bound in Khasminskii's estimate [58, (3.54)] does not depend on the initial point $(x, y) \in \hat{K}$ but only on the distance of the two points, we can integrate over the stationary distribution $\rho \times \rho$ on $\hat{M}$ and conclude that there is a $\gamma > 0$ such that

$$\frac{1}{t} \int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s, \cdot, x), \varphi(s, \cdot, y)) \in B_\gamma(\Delta) \cap \hat{K} \right) (\rho \times \rho)(\mathrm{d}x, \mathrm{d}y) \, \mathrm{d}s < \varepsilon/3$$

for any $t > 0$. Observe that by ergodicity

$$\frac{1}{t} \int_0^t \mathbb{P}\left((\varphi(s, \cdot, x), \varphi(s, \cdot, y)) \in \hat{M} \setminus \hat{K} \right) \mathrm{d}s \xrightarrow{t \to \infty} (\rho \times \rho)(\hat{M} \setminus \hat{K}) < \varepsilon/3$$

for $(\rho \times \rho)$-almost all $(x, y) \in K^2$. Since K^2 is compact, we can conclude that there is a $t^* > 0$ such that for all $t \geq t^*$

$$\frac{1}{t} \int_0^t \int_{\hat{K}} \mathbb{P}\left((\varphi(s, \cdot, x), \varphi(s, \cdot, y)) \in \hat{M} \setminus \hat{K} \right) (\rho \times \rho)(\mathrm{d}x, \mathrm{d}y) \, \mathrm{d}s < \varepsilon/3.$$

We define $K_1 = K^2 \cap \{(x, y) \in \hat{M} \ : \ d(x, y) \geq \gamma\}$ which is obviously compact. Let $\{P_t \ : \ t \geq 0\}$ denote the Markov semi-group acting on $C(M \times M)$ and $\{P_t^* \ : \ t \geq 0\}$ the adjoint semi-group acting on measures on $M \times M$. Observe that by the choice of K and K_1 and the above it follows for $t \geq t^*$ that

$$\begin{aligned}
&\frac{1}{t} \int_0^t \int_{\hat{M}} P_s \mathbb{1}_{\hat{M} \setminus K_1} \, \mathrm{d}(\rho \times \rho) \mathrm{d}s \\
&\qquad = \frac{1}{t} \int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s, \cdot, x), \varphi(s, \cdot, y)) \in B_\gamma(\Delta) \cap \hat{K} \right) (\rho \times \rho)(\mathrm{d}x, \mathrm{d}y) \, \mathrm{d}s \\
&\qquad + \frac{1}{t} \int_0^t \int_{\hat{M}} \mathbb{P}\left((\varphi(s, \cdot, x), \varphi(s, \cdot, y)) \in \hat{M} \setminus \hat{K} \right) (\rho \times \rho)(\mathrm{d}x, \mathrm{d}y) \, \mathrm{d}s \\
&\qquad < \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon.
\end{aligned}$$

Hence, for $t \geq t^*$ we obtain

$$\left(\frac{1}{t} \int_0^t P_s^*(\rho \times \rho) \mathrm{d}s \right) (K_1) = \frac{1}{t} \int_0^t \int_{\hat{M}} P_s \mathbb{1}_{K_1} \, \mathrm{d}(\rho \times \rho) \, \mathrm{d}s > 1 - \varepsilon.$$

Since $K_1 \subset \hat{M}$ compact, we have shown that the family of probability measures

$$\left\{ \frac{1}{t} \int_0^t P_s^*(\rho \times \rho) \, \mathrm{d}s \right\}_{t \geq t^*},$$

is uniformly tight on $\hat{M}$. Further note that these probability measures are supported on $\hat{M}$ due to absolute continuity of ρ. Hence, by Prokhorov's Theorem there is a probability measure ν on $\hat{M}$ and a time sequence $\{t_n\}_{n\in\mathbb{N}}$ such that $\frac{1}{t_n}\int_0^{t_n} P_s^*(\rho\times\rho)\mathrm{d}s \to \nu$ as $n\to\infty$. We know from [10, Proposition 2.6] that $P_t^*(\rho\times\rho)$ converges weakly to $\bar{\mu}$ as $t\to\infty$. So we can conclude that $\nu=\bar{\mu}$ and therefore $\bar{\mu}(\hat{M})=1$ as required. □

3.2.2 Pesin's formula

In this section, we want to investigate what positive Lyapunov exponents imply for the entropy of the system. We will follow [15, 73, 69, 59] and try to apply their work to situations interesting for stochastic bifurcation theory.

3.2.3 Entropy for discrete time systems

Firstly, we formulate the statements for $\mathbb{R}^d$ and random dynamical systems in discrete time generated by composed maps $\left\{f_\omega^n : n\geq 0, \omega\in\left(\Omega^{\mathbb{N}},\mathcal{B}(\Omega)^{\mathbb{N}},\nu^{\mathbb{N}}\right)\right\}$ which will be referred to as $\mathcal{X}^+(\mathbb{R}^d,\nu)$. Here, Ω denotes the set of two-times differentiable diffeomorphisms on $\mathbb{R}^d$ with the topology induced by uniform convergence on compact sets for all derivatives up to order 2. The maps are i.i.d. with law ν, and for a sequence $\omega=(f_0(\omega),f_1(\omega),\dots)\in\Omega^{\mathbb{N}}$ the compositions are given as

$$f_\omega^0=\mathrm{id}\,,\quad f_\omega^n=f_{n-1}(\omega)\circ f_{n-2}(\omega)\circ\cdots\circ f_0(\omega)\,.$$

Later we will formulate the statements for stochastic flows that are related to discrete time random dynamical systems via their time-one maps.

Definition 3.2.2 (Stationary measure)**.** A Borel probability measure μ on $\mathbb{R}^d$ is called a stationary measure of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ if

$$\mu(\cdot)=\int_\Omega \mu\left(f^{-1}(\cdot)\right)\nu(\mathrm{d}f)\,.$$

If ξ is a finite partition of a Lebesgue space $(X,\mathcal{B},\mu)$ and $C_1,\dots,C_k$ denote the elements of ξ, we define the *entropy* of ξ with respect to μ by

$$H_\mu(\xi)=-\sum_{j=1}^k \mu(C_j)\ln(\mu(C_j))\,.$$

Furthermore, for two partitions ξ_1 and ξ_2 we define

$$\xi_1\vee\xi_2=\{A\cap B : A\in\xi_1, B\in\xi_2\}\,,$$

such that elements of $\bigvee_{i=0}^{n-1}(f_\omega^i)^{-1}\xi$ are of the form

$$\{x : x\in C_{j_0}, f_\omega x\in C_{j_1},\dots,f_\omega^{n-1}x\in C_{j_{n-1}}\}$$

for some $(j_0, \ldots, j_{n-1})$ sometimes called the address of the orbit.

Following [73] and [15] we define the entropy of a random dynamical system in the following way:

Definition and Lemma 3.2.3 (Entropy)**.** For any finite partition ξ of $\mathbb{R}^d$ and stationary measure μ of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ the limit

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right) := \lim_{n\to\infty}\frac{1}{n}\int_{\Omega^{\mathbb{N}}} H_\mu\left(\bigvee_{k=0}^{n-1}(f_\omega^k)^{-1}\xi\right)\nu^{\mathbb{N}}(\mathrm{d}\omega)$$

exists. The number $h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right)$ is called the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ with respect to ξ. The number

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right) := \sup_\xi h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right)$$

is called the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$.

Consider the product spaces $\Omega^{\mathbb{N}}\times\mathbb{R}^d$ and $\Omega^{\mathbb{Z}}\times\mathbb{R}^d$. Since Ω equipped with the uniform topology on compact sets is a separable Banach space, the product σ-algebras $\mathcal{B}(\Omega)^{\mathbb{N}}\otimes\mathcal{B}(\mathbb{R}^d)$ and $\mathcal{B}(\Omega)^{\mathbb{Z}}\otimes\mathcal{B}(\mathbb{R}^d)$ satisfy

$$\mathcal{B}(\Omega)^{\mathbb{N}}\otimes\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\Omega^{\mathbb{N}}\times\mathbb{R}^d)\,,$$
$$\mathcal{B}(\Omega)^{\mathbb{Z}}\otimes\mathcal{B}(\mathbb{R}^d) = \mathcal{B}(\Omega^{\mathbb{Z}}\times\mathbb{R}^d)\,.$$

We denote the left shift operator on $\Omega^{\mathbb{N}}$ and $\Omega^{\mathbb{Z}}$ by τ, i.e.

$$f_n(\tau\omega) = f_{n+1}(\omega)$$

for all $\omega\in\Omega^{\mathbb{N}}$, $n\in\mathbb{N}$ and $\omega\in\Omega^{\mathbb{Z}}$, $n\in\mathbb{Z}$ respectively, and the associated skew product systems by

$$F:\Omega^{\mathbb{N}}\times\mathbb{R}^d\to\Omega^{\mathbb{N}}\times\mathbb{R}^d\,,\quad(\omega,x)\to(\tau\omega,f_0(\omega)x)\,,$$
$$G:\Omega^{\mathbb{Z}}\times\mathbb{R}^d\to\Omega^{\mathbb{Z}}\times\mathbb{R}^d\,,\quad(\omega,x)\to(\tau\omega,f_0(\omega)x).$$

First recall the following classical result which we already mentioned in Chapter 1 for continuous time systems.

Proposition 3.2.4. *Let μ be a probability measure on $\mathbb{R}^d$. Then μ is a stationary measure for $\mathcal{X}^+(\mathbb{R}^d,\nu)$ iff $\nu^{\mathbb{N}}\times\mu$ is an invariant measure for the one-sided skew product system F.*

Proof. See for example [59, Lemma I.2.3]. □

Furthermore, we have the following proposition which associates the invariant probability measure $\nu^{\mathbb{N}}\times\mu$ on $\Omega^{\mathbb{N}}\times\mathbb{R}^d$ to an invariant probability measure μ^* on $\Omega^{\mathbb{Z}}\times\mathbb{R}^d$.

Proposition 3.2.5. *For every stationary probability measure μ of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ there exists a unique Borel probability measure μ^* on $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ such that $G\mu^* = \mu^*$ and $P\mu^* = \nu^{\mathbb{N}} \times \mu$, where P denotes the projection to the measures on $\Omega^{\mathbb{N}} \times \mathbb{R}^d$.*

Proof. See [73, Proposition I.1.2]. □

Hence, we have established a relationship between the one-sided and two-sided time system with respect to their invariant measures. We can also establish a relation in terms of entropy; for that we need to study conditional entropies with respect to the appropriate σ-algebras and their generating partitions. We define the following σ-algebras with their corresponding generating partitions. For $\Omega^{\mathbb{N}} \times \mathbb{R}^d$ we have

$$\sigma_0 := \left\{\Gamma \times \mathbb{R}^d \,:\, \Gamma \in \mathcal{B}(\Omega^{\mathbb{N}})\right\}, \quad \xi_0 := \{\{\omega\} \times \mathbb{R}^d \,:\, \omega \in \Omega^{\mathbb{N}}\},$$

and for $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ we define

$$\sigma^+ := \left\{\left(\prod_{-\infty}^{-1} \Omega\right) \times \Gamma \times \mathbb{R}^d \,:\, \Gamma \in \mathcal{B}\left(\prod_{0}^{\infty} \Omega\right)\right\},$$
$$\xi^+ := \left\{\left(\prod_{-\infty}^{-1} \Omega\right) \times \{\omega\} \times \mathbb{R}^d \,:\, \omega \in \left(\prod_{0}^{\infty} \Omega\right)\right\},$$

and

$$\sigma := \left\{\Gamma' \times \mathbb{R}^d \,:\, \Gamma' \in \mathcal{B}(\Omega^{\mathbb{Z}})\right\}, \quad \xi := \{\{\omega\} \times \mathbb{R}^d \,:\, \omega \in \Omega^{\mathbb{Z}}\}.$$

Generally, for a probability space $(X, \mathcal{B}, \mu)$, a σ-algebra $\mathcal{A} \subset \mathcal{B}$ and a measurable partition ζ of X we define the corresponding conditional entropy as

$$H_\mu(\zeta|\mathcal{A}) := -\int_X \sum_{c \in \zeta} \mu(C|\mathcal{A}) \ln \mu(C|\mathcal{A}) \, \mathrm{d}\mu.$$

Following [59, 73, Section 0.4 and Section 0.5], we obtain:

Definition and Lemma 3.2.6. Consider a measure-preserving transformation $T : X \to X$ and a σ-algebra $\mathcal{A} \subset \mathcal{B}$ with $T^{-1}\mathcal{A} \subset \mathcal{A}$. Then for any measurable partition ζ of X with $H_\mu(\zeta|\mathcal{A}) < \infty$ the limit

$$h_\mu^{\mathcal{A}}(T, \zeta) := \lim_{n\to\infty} \frac{1}{n} H_\mu\left(\bigvee_{k=0}^{n-1} T^{-k}\zeta \middle| \mathcal{A}\right)$$

exists. The number $h_\mu^{\mathcal{A}}(T, \zeta)$ is called the $\mathcal{A}$-conditional entropy of T with respect to ζ. The number

$$h_\mu^{\mathcal{A}}(T) := \sup_\zeta h_\mu^{\mathcal{A}}(T, \zeta)$$

is called the $\mathcal{A}$-entropy of T.

We are now ready to state the very important theorem about the equality of entropy of the random dynamical system and the conditional entropies of the skew product systems, conditioned on the σ-algebras introduced above.

Theorem 3.2.7. *Let μ be a stationary probability measure of $\mathcal{X}^+(\mathbb{R}^d,\nu)$. Then the following equalities hold for the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$ and the conditional entropies of the skew product systems:*

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right) = h^{\sigma_0}_{\nu^{\mathbb{N}}\times\mu}(F) = h^{\sigma^+}_{\mu^*}(G) = h^{\sigma}_{\mu^*}(G)\,.$$

Proof. See [73, Proposition I.2.2 and I.2.3] □

One can define the entropies conditioned on the σ-algebras σ_0, σ^+ and σ also via the corresponding generating partitions ξ_0, ξ^+ and ξ. For details we refer to [15, Section 2]. It is important to state the relation between the σ-algebras and the associated partitions, since this is the key to proving Pesin's formula. It is also crucial for making the following observation: We can define

$$h_\mu(f_\omega,\zeta) := \lim_{n\to\infty}\frac{1}{n}H_\mu\left(\bigvee_{k=0}^{n-1}(f^k_\omega)^{-1}\zeta\right)\,,$$

in case the limit exists. Then according to [69, Proposition 2.1.2] we have for every partition ζ with $H_\mu(\zeta)<\infty$ that for almost all $\omega\in\Omega^{\mathbb{Z}}$

$$h_\mu(f_\omega,\zeta) = h^{\sigma}_{\mu^*}\left(G,\tilde{\zeta}\right)\,,$$

where $\tilde{\zeta} = \{\Omega^{\mathbb{Z}}\times A \,:\, A\in\zeta\}$. In particular, we obtain

$$\sup_\zeta h_\mu(f_\omega,\zeta) = h^{\sigma}_{\mu^*}(G)\ .$$

Summarising, we observe that averaged, conditioned and fibrewise entropy in the ways defined above are all the same quantity.

Furthermore, we recall Oseledets' Multiplicative Ergodic Theorem 2.1.1 and apply it to (G,μ^*) without μ^* being necessarily ergodic, following [69, Section 2.2]. There is an Oseledets splitting

$$T_xM = E_1(\omega,x)\oplus\cdots\oplus E_{p(\omega,x)}(\omega,x)$$

such that for μ^*-a.e. (ω,x)

$$\lim_{n\to\infty}\frac{1}{n}\ln\|\mathrm{D}f^{\pm n}_\omega v\| = \pm\lambda_i(\omega,x)\quad\text{if } 0\neq v\in E_i(\omega,x)\,. \tag{3.2.3}$$

The maps $(\omega,x)\mapsto p(\omega,x),\lambda_i(\omega,x),\dim E_i(\omega,x)$ are measurable and constant along orbits of G. In fact, there are functions $p,\lambda_i,d_i : M\to\mathbb{R}$ such that for μ^*-a.e. (ω,x)

$$p(\omega,x)=p(x),\ \ \lambda_i(\omega,x)=\lambda_i(x)\ \text{and}\ \dim E_i(\omega,x)=d_i\,,$$

where d_i is the multiplicity of λ_i. As before, if μ^* is ergodic, these functions are constant, i.e. the x-dependence vanishes.

In [15] we find five integrability assumptions which we denote by (A1)-(A5). It is not necessary for our purposes to list these here specifically as the assumptions are satisfied for sufficiently regular stochastic flows anyway. The theorem confirming Pesin's formula in this setting reads as follows:

Theorem 3.2.8 (Pesin's formula)**.** *Let $\mathcal{X}^+(\mathbb{R}^d, \nu)$ be a random dynamical system which has an absolutely continuous stationary probability measure μ and satisfies (A1)-(A5). Then we have*

$$h_\mu\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \mu(\mathrm{d}x)\,, \tag{3.2.4}$$

where $\lambda_i(x)^+$ are the positive Lyapunov exponents and $m_i(x)$ their multiplicities.

Proof. See [15]. □

3.2.4 Entropy for stochastic flows

We now make this result applicable to random dynamical systems induced by stochastic differential equations. In [15] the results are stated for general stochastic differential equations driven by semimartingales. Since all our models work with Brownian motion and time-independent vector fields, we transfer the general statements into this particular setting.

Consider the Itô SDE

$$\mathrm{d}X_t = F(X_t)\mathrm{d}t + G(X_t)\mathrm{d}W_t\,, \tag{3.2.5}$$

where $F : \mathbb{R}^d \to \mathbb{R}^d$ and $G : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are the drift and diffusion coefficients of the SDE and W_t is an m-dimensional Wiener process on the canonical filtered probability space of continuous paths $\left(\bar\Omega, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0}, \mathbb{P}\right)$ as introduced in Chapter 1. We further define the diffusion tensor

$$D_{ij}(x) = \sum_{k=1}^m G_{ik}(x)G_{jk}(x)\,.$$

Note that for the models in Chapter **??** and **??** Stratonovich and Itô noise are the same. In general, one has to account for the Itô-Stratonovich correction if one wants to relate the following results to results stated in Stratonovich form.

Assume that the entries of the diffusion matrix D and the drift F are in $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 1, 0 < \delta \leq 1$ and satisfy a typical linear growth condition such that the SDE induces a C^k random dynamical system (θ, φ) in the sense of Chapter 1. We relate the system to a stochastic flow of C^k diffeomorphisms in the sense of [63] by defining the maps

$$\tilde\varphi : \mathbb{R}^+_0 \times \mathbb{R}^+_0 \times \bar\Omega \times \mathbb{R}^d \to \mathbb{R}^d, \quad \tilde\varphi_{s,t}(\bar\omega, x) = \varphi(t-s, \theta_s\bar\omega, x)\,.$$

By this definition $\tilde\varphi_{s,t}(\bar\omega, \cdot)$ is a C^k diffeomorphism for each $s, t \geq 0$ and $\bar\omega \in \bar\Omega$.

We can associate the stochastic flow $\tilde\varphi$ with the probability space $\left(\tilde\Omega, \tilde{\mathcal{F}}, \tilde{\mathbb{P}}\right)$, where

$$\tilde\Omega = C_0\left(\mathbb{R}, C(\mathbb{R}^d, \mathbb{R}^d)\right) := \left\{f \,:\, \mathbb{R} \to C(\mathbb{R}^d, \mathbb{R}^d) \,:\, f \text{ is continuous and } f(0) = 0\right\}$$

is equipped with the topology of uniform convergence on compact sets and $\tilde{\mathcal{F}} := \mathcal{B}(\tilde{\Omega})$. The measure $\tilde{\mathbb{P}}$ is defined by $\tilde{\mathbb{P}}(\tilde{\omega}(0) = \mathrm{Id}) = 1$ and the property that for all $n \geq 0, 0 \leq t_1 < t_2 < \cdots < t_n$ and all $B \in \mathcal{B}(C(\mathbb{R}^d, \mathbb{R}^d))^{\otimes n}$ we have

$$\begin{aligned}&\tilde{\mathbb{P}}\left((\tilde{\omega}(t_1), \tilde{\omega}(t_2) \circ \tilde{\omega}(t_1)^{-1}, \ldots, \tilde{\omega}(t_n) \circ \tilde{\omega}(t_{n-1})^{-1}) \in B\right) \\ &= \mathbb{P}\left((\tilde{\varphi}_{t_0,t_1}, \tilde{\varphi}_{t_1,t_2}, \ldots, \tilde{\varphi}_{t_{n-1},t_n}) \in B\right) .\end{aligned}$$

Now let $k \geq 2$ and define Ω as above as the space of C^2 diffeomorphisms equipped with the uniform topology on compact sets. In this case, the measure

$$\nu(\cdot) = \mathbb{P}\{\bar{\omega} \in \bar{\Omega} \,:\, \tilde{\varphi}_{0,1}(\bar{\omega}, \cdot) \in \cdot\} \tag{3.2.6}$$

on $(\Omega, \mathcal{B}(\Omega))$ and the random diffeomorphisms

$$f_0(\omega) = \tilde{\omega}(1) = \tilde{\varphi}_{0,1}(\bar{\omega}, \cdot) = \varphi(1, \bar{\omega}, \cdot) \tag{3.2.7}$$

generate, as before, a random dynamical system in discrete time

$$\mathcal{X}^+(\mathbb{R}^d, \nu) = \left\{ f_\omega^n \,:\, n \geq 0, \omega \in \left(\Omega^{\mathbb{N}}, \mathcal{B}(\Omega)^{\mathbb{N}}, \nu^{\mathbb{N}}\right) \right\} .$$

Observe that the measure μ is stationary for this system if for any set $A \in \mathcal{B}(\mathbb{R}^d)$

$$\mu(A) = \int_{\bar{\Omega}} \mu\left((\tilde{\varphi}_{0,1}(\bar{\omega}, \cdot))^{-1}(A)\right) \mathbb{P}(\mathrm{d}\bar{\omega}) .$$

Let $P(t, x, \cdot)$ denote the transition probabilities associated to the stochastic differential equation. Then we make the following observation:

Lemma 3.2.9. *Any invariant probability measure ρ for the Markov semi-group associated to the stochastic differential equation* (3.2.5) *is stationary for the induced discrete time system $\mathcal{X}^+(\mathbb{R}^d, \nu)$.*

Proof. For all $A \in \mathcal{B}(\mathbb{R}^d)$ we have with Fubini that

$$\begin{aligned}\rho(A) &= \int_{\mathbb{R}^d} P(1, x, A)\, \rho(\mathrm{d}x) = \int_{\mathbb{R}^d} \int_{\bar{\Omega}} \mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega}, x))\, \mathbb{P}(\mathrm{d}\bar{\omega}) \rho(\mathrm{d}x) \\ &= \int_{\bar{\Omega}} \int_{\mathbb{R}^d} \mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega}, x))\, \rho(\mathrm{d}x) \mathbb{P}(\mathrm{d}\bar{\omega}) = \int_{\bar{\Omega}} \rho\left((\tilde{\varphi}_{0,1}(\bar{\omega}, \cdot))^{-1}(A)\right) \mathbb{P}(\mathrm{d}\bar{\omega}) ,\end{aligned}$$

which shows the claim. □

According to [63, Section 4.1], if D and F are $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 1$ and $0 < \delta \leq 1$ and the correction term

$$c(x, t) := \sum_{j=1}^{d} \frac{\partial D_{\cdot,j}}{\partial x_j}(x) \tag{3.2.8}$$

is also $C^{k,\delta}_{\mathrm{loc}}$, then the backward flow $\{\tilde{\varphi}_{t,s} \,:\, 0 \leq s \leq t < \infty\}$ is also $C^{k,\delta}_{\mathrm{loc}}$.

Using the relations between stochastic flows, continuous-time random dynamical systems and discrete time random dynamical systems as explained above, we can now formulate Pesin's formula for random dynamical systems induced by stochastic differential equations:

Theorem 3.2.10 (Pesin's formula for SDEs)**.** *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation with diffusion matrix D, drift F and correction term c, as given in* (3.2.8)*, all being $C^{k,\delta}_{\mathrm{loc}}$ for some $k \geq 2$. Let further be ρ an absolutely continuous stationary probability measure satisfying*

$$\int_{\mathbb{R}^d} (\ln(|x|+1)^{1/2} \, \rho(\mathrm{d}x) < \infty \,. \tag{3.2.9}$$

Then the discrete time random dynamical system $\mathcal{X}^+(\mathbb{R}^d, \nu)$ associated with (θ, φ) satisfies (A1)-(A5) and, hence,

$$h_\rho \left(\mathcal{X}^+(\mathbb{R}^d, \nu) \right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \rho(\mathrm{d}x)$$

holds.

Proof. A direct consequence of [15, Theorem 9.1]. □

We are now in the situation to apply this result to the models we have discussed in the previous chapters and obtain a more profound notion of chaos.

Corollary 3.2.11. *Let $X = \mathbb{R} \times \mathbb{S}^1$ and $\mathcal{X}^+(X, \nu)$ denote the discrete-time random dynamical system induced by model (**??**) in the way explained above. Let further be ρ the stationary probability measure for the SDE* (**??**) *and f be chosen as in* (**??**)*. If $\sigma > \sigma_0(\alpha, b)$, then the measure-theoretic entropy of $\mathcal{X}^+(X, \nu)$ is positive, i.e.*

$$h_\rho \left(\mathcal{X}^+(X, \nu) \right) > 0 \,.$$

Proof. The whole proof of Theorem 3.2.10 carries obviously over to $X \subset \mathbb{R}^d$. Condition 3.2.9 is satisfied due to the same considerations as in Section **??** concerning the boundedness by the Ornstein-Uhlenbeck process, i.e. the exponential decay of ρ in the amplitude.

The drift is smooth. The diffusion matrix D is given by

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} ,$$

and therefore $c(x,t) = 0$ for all x, t. Hence, all the conditions of Theorem 3.2.10 are satisfied. We further have for ρ-almost all x, $\mathbb{P}$-almost all $\bar{\omega}$ and $\nu^{\mathbb{N}}$-almost all ω that for all $0 \neq v \in X \backslash V_2(\bar{\omega}, x)$

$$\lambda_1 = \lim_{n \to \infty} \frac{1}{n} \| D_x f^n_\omega(x) v \| = \lim_{n \to \infty} \frac{1}{n} \| \mathrm{D}\varphi(n, \bar{\omega}, x) v \| > 0$$

according to Theorem **??**. Hence, the claim follows. □

Remark 3.2.12. Since ρ is ergodic and λ_1 is the only positive exponent, we actually have that

$$h_\rho\left(\mathcal{X}^+(X,\nu)\right) = \lambda_1 \,.$$

In other words, the entropy of the random dynamical system is identical to the first Lyapunov exponent.

Remark 3.2.13. The analogous corollary can obviously be stated for system (**??**) once Conjecture **??** is shown. The conditions of Theorem 3.2.10 are easily satisfied in this case.

3.2.5 SRB measures

Another more profound notion of chaos could be given by showing the SRB-property of the random measures μ_ω, which are the disintegrations of an invariant probability measure μ^* for the two-sided skew product system, i.e. $\mu^*(\mathrm{d}x, \mathrm{d}\omega) = \mu_\omega(\mathrm{d}x)\nu^{\mathbb{Z}}(\mathrm{d}\omega)$. Let's assume we are in exactly the same setting of a discrete time random dynamical system as above with the only difference that the state space is a compact smooth manifold M, calling such a system $\mathcal{X}^+(M,\nu)$, generated by C^2 diffeomorphisms and a law ν. Let the sample measures μ_ω be associated with a stationary measure μ in the sense of Proposition 3.2.5 and write $E_i(\omega,x)$ for the Oseledets spaces corresponding with the Lyapunov exponents $\lambda_i(x)$. We follow [69] for the following definitions and results.

Definition 3.2.14. Let $(\omega,x) \in \Omega^{\mathbb{Z}} \times M$ be s.t. $\lambda_i(x) > 0$ for some i. Then the unstable manifold and the stable manifold of the skew product flow G at (ω,x) are given by

$$W^u(\omega,x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{-n}x, f_\omega^{-n}y) < 0 \right\} \,,$$
$$W^s(\omega,x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{n}x, f_\omega^{n}y) < 0 \right\} \,.$$

At μ^*-a.e. (ω,x) with $\lambda_i(x) > 0$ for some i, $W^u(\omega,x)$ is a $\left(\sum_{\lambda_i>0} \dim E_i(\omega,x)\right)$-dimensional C^2 immersed submanifold of M. We set $W^u(\omega,x) = \{x\}$ if $\lambda_i(x) \le 0$ for all i. If η is a partition of $\Omega^{\mathbb{Z}} \times M$, η_ω denotes the restriction of η to the fibre $\{\omega\} \times M$ which is a partition of M. We write $\eta_\omega(x)$ for the element of η_ω that contains x.

Definition 3.2.15. A measurable partition η of $\Omega^{\mathbb{Z}} \times M$ is called subordinate to W^u if for μ^*-a.e. (ω,x), $\eta_\omega(x) \subset W^u(\omega,x)$ and contains an open neighbourhood of x in $W^u(\omega,x)$, this neighbourhood being taken in the submanifold topology of $W^u(\omega,x)$.

Identifying σ-algebras with their generating partitions, recall that σ is the partition of $\Omega^{\mathbb{Z}} \times M$ into sets of the form $\{\omega\} \times M$. If η is a partition subordinate to W^u, μ^* disintegrates into a system of conditional measures on elements of $\eta \vee \sigma$, denoted by $\left\{\mu^{*\eta\vee\sigma}_{(\omega,x)}\right\}$. For μ^*-a.e. (ω,x) we have the identification $\mu^{*\eta\vee\sigma}_{(\omega,x)} = (\mu_\omega)^{\eta_\omega}_x$. Finally let $\lambda_{W^u(\omega,x)}$ denote the Riemannian measure on $W^u(\omega,x)$.

Definition 3.2.16 (SRB measures)**.** The sample measures μ_ω are called SRB measures or absolutely continuous conditional measures on W^u-manifolds if for every measurable partition η subordinate to W^u, $\mu^{*\eta\vee\sigma}_{(\omega,x)}$ is absolutely continuous with respect to $\lambda_{W^u(\omega,x)}$ for μ^*-a.e. (ω, x).

Ledrappier & Young [69] can then prove the following statement.

Theorem 3.2.17. *Suppose the stationary measure μ of the random dynamical system $\mathcal{X}^+(M,\nu)$ is absolutely continuous with respect to the Lebesgue measure and $\lambda_1 > 0$. Then the sample measures μ_ω are SRB measures.*

Similarly to before, we can then formulate the following corollary for stochastic differential equations. As usual for the manifold case, we use the Stratonovich integral due to its classical properties in terms of the chain rule:

Corollary 3.2.18. *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation of type 3.2.1 with C^2 coefficients and stationary absolutely continuous probability distribution ρ on a compact manifold M. Let further $\lambda_1 > 0$ and $\tilde{\mu}$ be the invariant probability measure of (θ, φ) corresponding to ρ. Then the sample measures $\tilde{\mu}_{\bar{\omega}}$ are SRB measures.*

Proof. By Lemma 3.2.9, ρ is stationary for the induced discrete time system $\mathcal{X}^+(M,\nu)$. Then the claim follows immediately from Theorem 3.2.17 if we can show that $\tilde{\mu}_{\bar{\omega}}$ are the disintegrations μ_ω of the invariant measure μ^* of $\mathcal{X}^+(M,\nu)$ associated to ρ. According to [69, Proposition 1.2.3], the probability measures μ_ω are given by

$$\mu_\omega = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

However, identifying ω and $\bar{\omega}$ via relation (3.2.7) we have seen in Section 1.5 that $\tilde{\mu}_{\bar{\omega}}$ satisfies

$$\tilde{\mu}_{\bar{\omega}} = \lim_{n\to\infty} \varphi(n, \theta_{-n}\bar{\omega}, \cdot)\rho = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

Hence, the claim follows. □

We would like to apply this theorem to our setting and generally extend it to the non-compact case. Heuristically, this isn't a problem at all for dissipative systems with compact random attractors as for example models (**??**) and (**??**). However, the proof of Theorem 3.2.17 is technically very involved and makes a lot of references to the deterministic case [66, 67]. Hence, a rigorous proof analogous to [69] would require a complete own chapter. Checkroun et al. [26, Appendix] state a Theorem analogous to Corollary 3.2.18 for stochastic differential equations on $\mathbb{R}^d$ with global random attractors, also based on the results in [69]. However, they do not give a rigorous proof accounting for the non-compactness of the state space either.

Ignoring the mentioned technical intricacies, the picture is as follows. The chaotic random (point) attractors A_ω as depicted in Figures **??** and **??** are the support of the sample measures μ_ω and thereby non-singular subsets (maybe the same) of (as) the closures of the unstable manifolds $W^u(\omega, x)$ for all $x \in A_\omega$. The measures μ_ω are absolutely continuous with respect to Lebesgue measure on $W^u(\omega, x)$ which indicates chaotic motion on the attractor A_ω.

$$\begin{pmatrix} \sigma f'(\vartheta)s_2 - s_1\sigma f'(\vartheta)s_1s_2 \\ -s_2\sigma f'(\vartheta)s_1s_2 \end{pmatrix} \circ \mathrm{d}W_t^1 .$$

Following Section 2.2 we use the Itô-Stratonovich formula to observe similarly to (2.2.3) that

$$\frac{1}{t}\ln r_t = \frac{1}{t}\int_0^t \left[h_A(s_\tau) + k_B(s_\tau)\right] \mathrm{d}\tau + \mathcal{O}(t^{-1/2}) , \tag{3.2.10}$$

where the functions h_A and k_B are given by

$$h_A(s) = \langle s, As\rangle = -\alpha s_1^2 + b s_1 s_2 ,$$
$$k_B(s) = \frac{1}{2}\langle (B + B^*)\, s, Bs\rangle - \langle s, Bs\rangle^2 = \frac{1}{2}\sigma^2 f'(\vartheta)^2 s_2^2 - \sigma^2 f'(\vartheta)^2 s_1^2 s_2^2 .$$

The Furstenberg–Khasminskii formula for the top Lyapunov exponent (see chapter 1) is given by

$$\lambda_1 = \int_{\mathbb{R}}\int_{[0,1]}\int_{\mathbb{S}^1} (h_A(s) + k_B(s))\, \rho(\mathrm{d}s, \mathrm{d}\vartheta, \mathrm{d}y), \tag{3.2.11}$$

where ρ is the joint invariant measure for the diffusion s on the unit circle and the processes ϑ and y induced by (**??**). Similarly to the calculations in [52], we change variables to $s = (\cos\phi, \sin\phi)$. Note that the functions h_A and k_B are π-periodic, which implies that the formula (3.2.11) for the top Lyapunov exponent reads as

$$\begin{aligned} \lambda_1 = \int_{\mathbb{R}\times[0,1]\times[0,\pi]} \Big(& -\alpha\cos^2\phi + b\cos\phi\sin\phi \\ & + \frac{1}{2}\sigma^2 f'(\vartheta)^2 \sin^2\phi(1 - 2\cos^2\phi)\Big)\tilde{\rho}(\mathrm{d}\phi, \mathrm{d}\vartheta, \mathrm{d}y), \end{aligned} \tag{3.2.12}$$

where $\tilde{\rho}$ denotes the corresponding image measure of ρ. The SDE determining the dynamics of $\phi \in [0, \pi)$ reads as

$$\mathrm{d}\phi = -\frac{1}{\sin\phi}\mathrm{d}s_1 = (\alpha\cos\phi\sin\phi + b\cos^2\phi)\mathrm{d}t - \sigma f'(\vartheta)\sin^2\phi \circ \mathrm{d}W_t^1 , \tag{3.2.13}$$

where we denote

$$c(\phi, \vartheta) = \sigma f'(\vartheta)\sin^2\phi \quad \text{and} \quad d(\phi) = \alpha\cos\phi\sin\phi + b\cos^2\phi . \tag{3.2.14}$$

The integrand in (3.2.10) and thereby in (3.2.12) only depends on ϕ and not on ϑ and y if $f'(\vartheta)^2$ is constant, and in addition to that, the dependence on ϑ in the Fokker–Planck equation for ϕ, is restricted to $f'(\vartheta)^2$. This means that the calculation of λ_1 becomes much simpler if $f'(\vartheta)^2$ is constant, an observation that we exploit in the following.

3.2.6 Explicit formula for the top Lyapunov exponent

We continue the analysis of the top Lyapunov exponent under the assumption that $f : [0,1] \to \mathbb{R}$ is given by (**??**). Importantly, $f'(\vartheta)^2$ is constant in this special case and our results hold in fact for every continuous and piecewise linear f with constant absolute value of the derivative almost everywhere.

The map is not differentiable at $\frac{1}{2}$ and 0, and we verify that does not cause any problems. We need the following results to justify the variational equation defining $\mathrm{D}\varphi$:

Lemma 3.2.19. *Let $W : \mathbb{R}_0^+ \times \Omega \to \mathbb{R}$ denote the canonical real-valued Wiener process, and let $X : \mathbb{R}_0^+ \times \Omega \to [0,1]$ be a stochastic process adapted to the natural filtration of the Wiener process. Furthermore, suppose there exists a measurable set $A \subset [0,1]$ such that*

$$\mathbb{P}\big(\big\{\omega \in \Omega : \textstyle\int_0^t \mathbb{1}_{\{X_u \in A\}}\,\mathrm{d}u = 0\big\}\big) = 1 \quad \textit{for all } t > 0\,, \tag{3.2.15}$$

i.e. A is visited only on a measure zero set with full probability. Consider a measurable function $g : [0,1] \to [0,1]$ such that $g = 0$ on $[0,1] \setminus A$. Then

$$\int_0^t g(X_u)\,\mathrm{d}W_u = 0 \quad \textit{almost surely for all } t > 0\,.$$

Proof. The statement follows directly from Itô's isometry

$$\begin{aligned}\mathbb{E}\left[\left(\int_0^t g(X_u)\mathrm{d}W_u\right)^2\right] &= \mathbb{E}\left[\int_0^t g(X_u)^2\mathrm{d}u\right] \\ &= \mathbb{E}\left[\int_0^t \left(g(X_u)^2\mathbb{1}_{\{X_u\in A\}} + g(X_u)^2\mathbb{1}_{\{X_u\in[0,1]\setminus A\}}\right)\mathrm{d}u\right] = 0\,,\end{aligned}$$

where the last equality follows immediately from (3.2.15) and $g = 0$ on $[0,1] \setminus A$. We conclude

$$\left(\int_0^t g(X_u)\,\mathrm{d}W_u\right)^2 = 0 \quad \text{almost surely}$$

due to nonnegativity, and the claim follows. □

Proposition 3.2.20. *Let f' denote the weak derivative of f as given by (**??**). Then the choice of representative of f' by determining $f'(\frac{1}{2})$ and $f'(0)$ does not affect the solution to the variational equation (**??**).*

Proof. First, we show that

$$\mathbb{P}\big(\big\{\omega \in \Omega : \textstyle\int_0^t \mathbb{1}_{\{\vartheta_u = 1/2\}}\,\mathrm{d}u = 0\big\}\big) = 1 \quad \text{for all } t > 0$$

by assuming the contrary to obtain a contradiction. As ϑ is a continuously differentiable process, this implies that $\vartheta_u = \frac{1}{2}$ for $u \in [t^*, t^* + \varepsilon]$ for some $t^* \in (0,t)$ and $\varepsilon > 0$ with positive probability. This leads to $y(u) = -\frac{1}{b} \mod 1$ for $u \in (t^*, t^* + \varepsilon)$ with positive probability. However, this implies that the continuous process y_u for $u \in (t^*, t^* + \varepsilon)$ given by

$$\mathrm{d}y = -\alpha y \,\mathrm{d}u + \sigma \,\mathrm{d}W_u$$

is constant with positive probability. This contradicts its definition as an Ornstein–Uhlenbeck process. The same reasoning obviously holds for $\theta = 0$.

Let $f_1' = f_2' = f'$ on $(0,1) \setminus \{\frac{1}{2}\}$ and assign arbitrary values at $\frac{1}{2}$ and 0. Define

$$\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma f_1'(\vartheta) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^1 \,,$$
$$dw = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} w \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma f_2'(\vartheta) \\ 0 & 0 \end{pmatrix} w \circ \mathrm{d}W_t^1 \,.$$

We apply Lemma 3.2.19 by choosing $X_u = \vartheta_u$ and $g(\vartheta_u) = f_1'(\vartheta_u) - f_2'(\vartheta_u)$ to conclude that

$$\int_0^t f_1'(\vartheta_u) \,\mathrm{d}W_u = \int_0^t f_2'(\vartheta_u) \,\mathrm{d}W_u \quad \text{almost surely.}$$

As we do not have an Itô–Stratonovich correction in this case, we can infer that $v_t = w_t$ almost surely for all $t > 0$. □

We view f' in the weak sense, disregarding the points $\frac{1}{2}$ and 0, and we define $f'(\vartheta) = \operatorname{sign}(\frac{1}{2} - \vartheta)$, where

$$\operatorname{sign}(x) = \begin{cases} 1 & \text{if } x \geq 0\,, \\ -1 & \text{if } x < 0\,. \end{cases}$$

By Proposition 3.2.20, $\mathrm{D}\varphi(t, \omega, x)$ does not depend on the choice of $f'(\frac{1}{2})$, so the variational equation (**??**) becomes

$$\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma \operatorname{sign}(\frac{1}{2} - \vartheta_t) \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^1 \,. \tag{3.2.16}$$

We derive the following formula for the first Lyapunov exponent in this case:

Proposition 3.2.21. *The top Lyapunov exponent of system (**??**) with f as defined in (**??**) is given by*

$$\lambda_1 = \int_0^\pi q(\phi) p(\phi) \,\mathrm{d}\phi \,, \tag{3.2.17}$$

where $q(\phi) := -\alpha \cos^2 \phi + b \cos \phi \sin \phi + \frac{1}{2}\sigma^2 (1 - 2\cos^2 \phi) \sin^2 \phi$, and $p(\phi)$ is the solution of the stationary Fokker–Planck equation $\mathcal{L}^* p = 0$. $\mathcal{L}^*$ is the formal L^2-adjoint of the generator $\mathcal{L}$, which is given by

$$\mathcal{L}g(\phi) = \left(d(\phi) + \frac{1}{2}\tilde{c}(\phi)\tilde{c}'(\phi) \right) g'(\phi) + \frac{1}{2}\tilde{c}^2(\phi) g''(\phi) \,, \tag{3.2.18}$$

where $d = d(\phi)$ is defined as in (3.2.14)*, and $\tilde{c}(\phi) := \sigma \sin^2 \phi$.*

Proof. Note that in our special case, the function c from (3.2.14) reads as

$$c(\phi,\vartheta) = \sigma\,\mathrm{sign}(\tfrac{1}{2}-\vartheta)\sin^2\phi\,,$$

which implies that both $c(\phi,\vartheta)c'(\phi,\vartheta)$ and $c^2(\phi,\vartheta)$ do not depend on ϑ and read as $\tilde{c}\tilde{c}'$ and $\tilde{c}^2$, respectively. Consider the SDE for the process $\phi(t)$ in Itô form

$$\mathrm{d}\phi = r(\phi)\mathrm{d}t + c(\phi,\vartheta)\mathrm{d}W_t\,,$$

where

$$r(\phi) = d(\phi) + \frac{1}{2}c(\phi,\vartheta)c'(\phi,\vartheta) = d(\phi) + \frac{1}{2}\tilde{c}(\phi)\tilde{c}'(\phi)\,.$$

As the coefficients of the SDE are smooth in ϕ, we consider the kinetic equation for the probability density function of the process $\phi(t)$ (cf. [86])

$$\frac{\partial p(\psi,t)}{\partial t} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n!}\frac{\partial^n}{\partial\psi^n}\left[a_n(\psi,t)p(\psi,t)\right],$$

where

$$a_n(\psi,t) = \lim_{\Delta t\to 0}\frac{1}{\Delta t}\mathbb{E}\left[(\phi(t+\Delta t)-\phi(t))^n|\phi(t)=\psi\right] \quad \text{for all } n\in\mathrm{N}\,.$$

Pick Δt small, denote $\Delta W_t = W(t+\Delta)-W(t)$ and recall that $\mathbb{E}[\Delta W_t]=0$ and $\mathbb{E}[(\Delta W_t)^2]=\Delta t$. Observe that

$$\phi(t+\Delta t)-\phi(t) = r(\phi(t))\Delta t + c(\phi(t),\vartheta(t))\Delta W_t + o(\Delta t)\,,$$

and

$$\begin{aligned}(\phi(t+\Delta t)-\phi(t))^2 &= r^2(\phi(t))(\Delta t)^2 + c^2(\phi(t),\vartheta(t))(\Delta W_t)^2\\ &\quad + 2r(\phi(t))c(\phi(t),\vartheta(t))\Delta W_t\Delta t + o(\Delta t)\,.\end{aligned}$$

Since ΔW_t is independent from $\phi(t)$ and $\vartheta(t)$, we obtain that

$$a_1(\psi,t) = r(\psi) \quad\text{and}\quad a_2(\psi,t) = \tilde{c}^2(\psi)\,.$$

We can see immediately from above that $a_n(\psi,t)=0$ for $n\geq 3$. This proves (3.2.18), and (3.2.17) follows from (3.2.12). □

In this case, the stationary Fokker–Planck equation reduces to a linear nonautonomous ordinary differential equation for $p=p(\phi)$ defined on $[0,\pi)$ with periodic boundary conditions:

$$-\left(\frac{1}{2}\tilde{c}^2p\right)' + \left(d+\frac{1}{2}\tilde{c}\tilde{c}'\right)p = \kappa,$$

where the constant κ has to be determined from the boundary and the normalization condition.

The ordinary differential equations is given in explicit form as

$$
\begin{aligned}
p' &= \left(\frac{2d(\phi)}{\tilde{c}^2(\phi)} - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right) p + \frac{2\kappa}{\tilde{c}^2(\phi)} \\
&= \left(2\frac{\alpha}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-1}\phi + 2\frac{b}{\sigma^2}\frac{1}{\sin^2\phi}\tan^{-2}\phi - \frac{\tilde{c}'(\phi)}{\tilde{c}(\phi)}\right) p + \frac{2\kappa}{\tilde{c}^2(\phi)} \,.
\end{aligned}
$$

The solution of this equation follows from the variation of constants formula, and is given by

$$
p(\phi) = \frac{G(\phi)\int_\phi^\pi \frac{2}{\tilde{c}^2(\psi)G(\psi)}\,\mathrm{d}\psi}{\int_0^\pi G(\phi)\int_\phi^\pi \frac{2}{\tilde{c}^2(\psi)G(\psi)}\,\mathrm{d}\psi\,\mathrm{d}\phi}\,, \tag{3.2.19}
$$

where

$$
G(\phi) = \frac{1}{\tilde{c}(\phi)}\exp\left(-\frac{1}{\sigma^2}\left[\alpha\tan^{-2}\phi + \frac{2}{3}b\tan^{-3}\phi\right]\right).
$$

The derivation of a closed formula for λ_1 and λ_2 using (3.2.19) for the stationary density of the process ϕ_t closely follows Imkeller and Lederer [52]. It can also be seen as a special case of the more general formulas obtained in [53].

Theorem 3.2.22 (Formula for λ_1 and λ_2)**.** *Consider the stochastic differential equation* (**??**)*, where the function f is of the form* (**??**)*. Then the two Lyapunov exponents are given by*

$$
\lambda_1(\alpha,b,\sigma) = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,, \tag{3.2.20}
$$

$$
\lambda_2(\alpha,b,\sigma) = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,. \tag{3.2.21}
$$

where

$$
m_{\sigma,b,\alpha}(v) = \frac{\frac{1}{\sqrt{v}}\exp\left(-\frac{\sigma^4 b^4}{6}v^3 + \frac{\alpha^2}{2}v\right)}{\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\frac{\sigma^4 b^4}{6}u^3 + \frac{\alpha^2}{2}u\right)\mathrm{d}u}. \tag{3.2.22}
$$

Proof. We define the function $g : [0,\phi) \to \mathbb{R}\cup\{\infty\}$ by

$$
\begin{aligned}
g(\phi) &:= -\ln\sin\phi \quad \text{for all } \phi\in(0,\pi)\,, \\
g(0) &:= \infty\,,
\end{aligned}
$$

and apply this function formally to the generator as given in (3.2.18):

$$
\begin{aligned}
\mathcal{L}g(\phi) &= \left(\alpha\cos\phi\sin\phi + b\cos^2\phi + \sigma^2\sin^3\phi\cos\phi\right)\left(-\tan^{-1}\phi\right) + \frac{1}{2}\sigma^2\sin^4\phi\frac{1}{\sin^2\phi} \\
&= -b\tan^{-1}\phi + q(\phi).
\end{aligned}
$$

This can be made precise by choosing suitable C^∞-functions to approximate g. Observe that

$$
0 = \int_0^\pi g\;\mathcal{L}^* p\;\mathrm{d}\phi = \int_0^\pi \mathcal{L}g\;p\;\mathrm{d}\phi = \int_0^\pi \left(-b\tan^{-1} + q\right)p\;\mathrm{d}\phi\,,
$$

and we conclude that

$$\lambda_1 = b\int_0^\pi \tan^{-1}(\phi)p(\phi)\mathrm{d}\phi\,.$$

Working with expression (3.2.19), we conduct a change of variables $s = \tan^{-1}\phi$ and $t = \tan^{-1}\psi$ which leads to

$$\lambda_1 = b\frac{\int_{-\infty}^{\infty}\int_{-\infty}^{s} s\exp\left(-\frac{1}{\sigma^2}\left[\alpha(s^2-t^2)+\frac{2}{3}b(s^3-t^3)\right]\right)\,\mathrm{d}t\,\mathrm{d}s}{\int_{-\infty}^{\infty}\int_{-\infty}^{s}\exp\left(-\frac{1}{\sigma^2}\left[\alpha(s^2-t^2)+\frac{2}{3}b(s^3-t^3)\right]\right)\,\mathrm{d}t\,\mathrm{d}s}\,. \tag{3.2.23}$$

We introduce a new variable $u = s - t$, which implies that $u \in (0,\infty)$. We observe

$$\begin{aligned}\alpha s^2 - \alpha(s-u)^2 + \frac{2}{3}bs^3 - \frac{2}{3}b(s-u)^3 &= -\alpha u^2 + 2\alpha su + 2bus^2 - 2bu^2s + \frac{2}{3}bu^3\\ &= 2bu\left(s - \frac{u-\alpha/b}{2}\right)^2 + \frac{b}{6}u^3 - \frac{1}{2}u\frac{\alpha^2}{b}\,.\end{aligned}$$

Using this expression, we modify (3.2.23) and obtain

$$\begin{aligned}\lambda_1 &= b\frac{\int_0^\infty\int_{-\infty}^{\infty} s\exp\left(-\frac{2u}{\sigma^2}\left(s-\frac{u-\alpha/b}{2}\right)^2\right)\mathrm{d}s\exp\left(-\frac{b}{6a^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\,\mathrm{d}u}{\int_0^\infty\int_{-\infty}^{\infty}\exp\left(-\frac{2bu}{\sigma^2}\left(s-\frac{u-\alpha/b}{2}\right)^2\right)\mathrm{d}s\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\,\mathrm{d}u}\\ &= b\frac{\int_0^\infty \frac{1}{\sqrt{u}}\frac{u-\alpha/b}{2}\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\,\mathrm{d}u}{\int_0^\infty\frac{1}{\sqrt{u}}\exp\left(-\frac{b}{6\sigma^2}u^3+\frac{1}{2}u\frac{\alpha^2}{\sigma^2 b}\right)\,\mathrm{d}u}\\ &= -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\frac{\int_0^\infty\frac{1}{\sqrt{v}}v\exp\left(-\frac{\sigma^4 b^4}{6}v^3+\frac{\alpha^2}{2}v\right)\mathrm{d}v}{\int_0^\infty\frac{1}{\sqrt{v}}\exp\left(-\frac{\sigma^4 b^4}{6}v^3+\frac{\alpha^2}{2}v\right)\mathrm{d}v}\,,\end{aligned}$$

where we have done another change of variables $v = u/(b\sigma^2)$ in the last equality, and we used well-known properties of the normal distribution. Hence, we write

$$\lambda_1 = -\frac{\alpha}{2} + \frac{b^2\sigma^2}{2}\int_0^\infty v\;m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,,$$

where $m_{\sigma,b,\alpha}(v)$ is given as in (3.2.22). From Proposition **??**, we obtain that $\lambda_1 + \lambda_2 = -\alpha$, and this means that

$$\lambda_2 = -\frac{\alpha}{2} - \frac{b^2\sigma^2}{2}\int_0^\infty v\;m_{\sigma,b,\alpha}(v)\mathrm{d}v.$$

This finishes the proof of this theorem. □

Chapter 4

Topological conjugacies and bifurcations

Chapter 5

Local RDS, quasi-stationary dynamics and bifurcations

Note that model (**??**) operates with unbounded white noise whereas the nature of the deterministic Hopf bifurcation is local, in the sense that the bifurcation happens in a neighbourhood of the origin and $\alpha = 0$. The same problem holds for the typical example of stochastic pitchfork bifurcation where the drift f_α is given as the derivative of a potential V_α, i.e.

$$f_\alpha(x) = -\partial_x V_\alpha(x)\,, \quad \text{with } V_\alpha = -\frac{\alpha}{2}x^2 + \frac{1}{4}x^4\,. \tag{5.0.1}$$

Without noise, the bifurcation implies a change of the attractor from $\{0\}$ for $\alpha \leq 0$ to $[-\sqrt{\alpha}, \sqrt{\alpha}]$ for $\alpha > 0$. Recall that the bifurcation is "destroyed" in the presence of noise in the sense that the random attractor is a random equilibrium for all $\sigma > 0, \alpha \in \mathbb{R}$ [32]. The white noise lets the system explore the whole state space and the global stability results in a negative Lyapunov exponent. Local finite-time instabilities can be captured by the dichotomy spectrum Σ which is given by $\Sigma_\alpha = [-\infty, \alpha]$ for all $\alpha \in \mathbb{R}$. However, the dichotomy spectrum is generally not as directly interpretable as the sign of a Lyapunov exponent and still contains a measure of global stability by covering $\mathbb{R}_0^-$. The question is what kind of analysis can most accurately describe a local stochastic bifurcation, in particular if the system does not exhibit global stability outside a critical neighbourhood. If the system is not in normal form, such a problem naturally arises for pitchfork as well as Hopf bifurcations.

We tackle this problem by embedding stochastic bifurcation theory into the context of Markov processes that induce a random dynamical system and are absorbed at the boundary of a domain. The process is said to be killed when it hits the trap and it is assumed that this happens almost surely at a finite hitting time T. We investigate the asymptotic dynamics of surviving trajectories. Due to the loss of mass by absorption at the boundary, the existence of a stationary distribution is impossible and, therefore, stationarity is replaced by quasi-stationarity. A *quasi-stationary distribution* preserves mass along the process conditioned on survival. Given a quasi-stationary distribution, one can derive the existence of a *quasi-ergodic distribution* for which expectations of time averages conditioned on survival equal the space average with respect to this distribution. We introduce these concepts and summarise important results, in particular for stochastic differential equations, following [25, 24, 28, 74], in Section 5.1.

In Section 5.2, we develop a theory of asymptotic average Lyapunov exponents for systems absorbed at the boundary of a domain. We mainly focus on stochastic differential equations with additive noise, i.e.

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,\ X_0 = \xi \in E\,,$$

where f is continuously differentiable and $E \subset \mathbb{R}^d$ a bounded domain. Using work by Villemonais, Champagnat, He and others [25, 24, 49], we are able to show Proposition 5.2.3 which says that the conditioned expectation of the finite-time Lyapunov exponents of a system induced by such an SDE converges to a real number λ. This *conditioned average Lyapunov exponent* is given by a Furstenberg-Khasminskii-formula, i.e. the average of a functional with respect to the quasi-ergodic distribution.

Furthermore, we can show in Theorem 5.2.8 that the finite-time Lyapunov exponents of the surviving trajectories converge to its assemble average λ in probability. Note that this gives λ the strongest possible dynamical meaning in the setting with absorption at the boundary since convergence almost surely is ruled out by the killing of almost all trajectories. The crucial ingredient for proving Theorem 5.2.8 is Lemma 5.2.7 which shows decay of correlations conditioned on survival.

Section 5.3 gives negative λ a dynamical interpretation. We prove the local synchronisation Theorem 5.3.1 which says that, if $\lambda < 0$, there is exponentially fast local synchronisation of trajectories in discrete time with arbitrarily high probability. We formulate the result for general differentiable random dynamical systems with killing and observe the implications for additive noise SDEs as a corollary.

In Section 5.4 we try to relate quasi-stationary and quasi-ergodic measures to sample measures of the killed random dynamical system. Leaving out the past of the system enables us to show Propositions 5.4.2 and 5.4.4 which establish a correspondence with conditionally invariant measures for the associated open system on the skew-product space where the hole is confined to the state space. In two-sided time it turns out to be unclear how one could relate quasi-stationary or quasi-ergodic distributions to conditionally invariant measures and in particular its sample or fibre measures. We discuss some ideas into this direction, as for example the survival process, but remain sceptical whether this is a feasible endeavour.

We further define the dichotomy spectrum for the situation with killing in Section **??** and prove in Theorem **??** that the essential supremum and infimum of finite-time Lyapunov exponents converge to the boundary of the dichotomy spectrum. In addition to that, we show that the dichotomy spectrum consists of a finite number $n \in \{1, \dots, d\}$ of closed intervals (Theorem **??**), where d is the dimension of the space.

In Section **??** we consider the examples of pitchfork and Hopf bifurcation with additive noise and approximate the quasi-ergodic distribution using a finite-difference scheme. We analyse the change of sign of λ and the corresponding bifurcation behaviour depending on the bifurcation parameter and the diameter of the domain. We also determine the dichotomy spectrum for the pitchfork problem with killing. Following considerations about the possibility of a Lyapunov spectrum in Section 5.2 we conduct numerical experiments for the Hopf example that indicate its existence. However, a proof seems out of reach.

In short, this chapter is structured as follows. Section 5.1 gives an overview over results concerning killed processes and quasi-stationary distributions which we need in the following. In Section 5.2 we prove the existence of a characteristic Lyapunov exponent λ for killed systems generated by additive noise SDEs and the fact that finite-time Lyapunov exponents converge to this quantity in conditioned probability. Section 5.3 discusses local synchronisation results for negative λ and Section 5.4 relates the developed theory of quasi-stationary dynamics to open systems on the skew product space. In Section **??** we transfer important results for exponential dichotomies of random dynamical systems to the situation with killing. Section **??** discusses the stochastic pitchfork and Hopf bifurcation in bounded domains with absorbing boundary, using the stability theory developed in the previous sections.

5.1 Quasi-stationary and quasi-ergodic distributions

5.1.1 General setting

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov process (see e.g. [80, Definition III.1.1]) on a topological state space E with boundary ∂E and Borel σ-algebra $\mathcal{E} := \mathcal{B}(E \cup \partial E)$, where the process is associated with a family of probabilities $(\mathbb{P}_x)_{x\in E}$ on a filtered space $(\Omega, (\mathcal{F}_t)_{t\geq 0})$. We have

$$\mathbb{P}_x(X_0 = x) = 1 \quad \text{for all } x \in E \cup \partial E$$

and the transition probabilities $(\hat{P}_t)_{t\geq 0}$ are given by

$$\hat{P}_t(x, A) = \mathbb{P}_x(X_t \in A) \quad \text{for all } x \in E \cup \partial E, A \in \mathcal{E}\,.$$

The process is further associated with a semi-group of operators $(P_t)_{t\geq 0}$ given by

$$P_t f(x) = \mathbb{E}_x[f(X_t)]$$

for $f \in \tilde{\mathcal{E}} := B(E \cup \partial E)$, the measurable and bounded functions from $E \cup \partial E$ to $\mathbb{R}$.

We consider the Markov process to be absorbed at ∂E, i.e. $X_s \in \partial E$ implies $X_t = X_s$ for all $t \geq s$. This implies that the random variable

$$T := \inf\{t \geq 0, X_t \in \partial E\}$$

is a stopping time and we let $X_t = X_T$ for all $t \geq T$. We make two assumptions that cover all the problems we are interested in: firstly, we assume for all $x \in E$ that

$$T < \infty \ \mathbb{P}_x - \text{a.s.}\,, \tag{5.1.1}$$

which means that almost every trajectory hits the boundary in finite time. Secondly, we demand that for all $x \in E$ and $t \geq 0$

$$\mathbb{P}_x(T > t) > 0\,, \tag{5.1.2}$$

i.e. that for any initial condition, the probability of survival until any given time is positive. We will further mainly consider Markov processes which induce a random dynamical system but we will specify this later.

Quasi-stationary distributions

For any measure μ on E we will use the notation

$$\mathbb{P}_\mu = \int_E \mathbb{P}_x \, \mu(\mathrm{d}x)\,.$$

Almost every statement in random dynamical system theory requires a stationary measure for the underlying Markov process $(X_t)_{t\geq 0}$ which is a measure μ on E with the property that

$$\mathbb{P}_\mu(X_t \in A) = \mu(A) \;\; \text{for all measurable } A \subset E\,, t \geq 0\,.$$

Note that in the situation with trapping at the boundary such a measure cannot exist: assume there was a stationary measure μ. Due to (5.1.1) there is a $t^*(\mu) > 0$ such that for all $t > t^*(\mu)$

$$1 > \int_E \mathbb{P}_x(T > t)\, \mu(\mathrm{d}x) = \mathbb{P}_\mu(X_t \in E) = \mu(E) = 1\,,$$

which is a contradiction. This leads to the following definition.

Definition 5.1.1 (QSD)**.** A *quasi-stationary distribution* (QSD) is a probability measure ν on E such that for all $t \geq 0$ and all measurable sets $B \subset E$

$$\mathbb{P}_\nu\left(X_t \in B | T > t\right) = \nu(B)\,. \tag{5.1.3}$$

Without stating any further assumptions, we can make the following well-known observation [46] about the exponentially distributed killing time of a process started with a QSD ν.

Proposition 5.1.2. *If ν is a QSD, then there exists a $\lambda_0 < 0$ such that for all $t \geq 0$*

$$\mathbb{P}_\nu(T > t) = e^{\lambda_0 t}\,,$$

that is, starting from ν, T is exponentially distributed with parameter λ_0. We call λ_0 the (exponential) survival rate and $-\lambda_0$ the (exponential) escape rate associated with the quasi-stationary distribution.

Proof. We follow [28]: from the definition of a QSD and a classical application of the Monotone-Class Theorem [80, Theorem II.3.1], we have for all measurable and bounded observables g :

$E \to \mathbb{R}$ that

$$\mathbb{E}_\nu(g(X_t)\mathbb{1}_{T>t}) = \left(\int_E g \,\mathrm{d}\mu\right) \mathbb{P}_\nu(T > t) \quad \text{for all } t \geq 0 \,.$$

Choosing $g(x) = \mathbb{P}_x(T > s)$ for some $s \geq 0$ we obtain

$$\mathbb{E}_\nu(\mathbb{P}_{X_t}(T > s)\mathbb{1}_{T>t}) = \mathbb{P}_\nu(T > s)\mathbb{P}_\nu(T > t) \quad \text{for all } t \geq 0 \,.$$

Using the Markov property and the commutation property of conditional expectations, we deduce for all $t, s \geq 0$ that

$$\begin{aligned}\mathbb{P}_\nu(T > t+s) &= \mathbb{E}_\nu(\mathbb{1}_{T>t+s}) = \mathbb{E}_\nu\left(\mathbb{E}(\mathbb{1}_{T>t+s}|\mathcal{F}_t)\mathbb{1}_{T>t}\right) \\ &= \mathbb{E}_\nu(\mathbb{E}_{X_t}(\mathbb{1}_{T>s})\mathbb{1}_{T>t}) = \mathbb{P}_\nu(T > s)\mathbb{P}_\nu(T > t) \,.\end{aligned}$$

Since the equality $f(t+s) = f(t)f(s)$ is only satisfied by exponential functions, the claim follows. □

Champagnat and Villemonais [25] have given three equivalent conditions for exponential convergence to a quasi-stationary distribution. We restrict ourselves to formulating the weakest assumption among them, denoted by (A') in the original paper. This condition will turn out to be satisfied by the stochastic differential equations we are investigating:

Assumption (A) There exists a family of probability measures $(\nu_{x_1,x_2})_{x_1,x_2\in E}$ on E such that

(A1) there exist $t_0, c_1 > 0$ such that for all $x_1, x_2 \in E$,

$$\mathbb{P}_{x_i}(X_{t_0} \in \cdot | T > t_0) \geq c_1 \nu_{x_1,x_2}(\cdot) \quad \text{for } i = 1, 2\,;$$

(A2) there exists $c_2 > 0$ such that for all $x_1, x_2 \in E$ and $t \geq 0$,

$$\mathbb{P}_{\nu_{x_1,x_2}}(T > t) \geq c_2 \sup_{x\in E} \mathbb{P}_x(T > t) \,.$$

We summarise three results from [25] in the following theorem which contains the most relevant ingredients for our further purposes. Statement (a) guarantees the existence of a QSD with exponential convergence of initial distributions. Part (b) characterizes the limit of the survival probability for an initial Dirac distribution at x divided by the survival probability under the QSD as the value at x of an eigenfunction of the generator $\mathcal{L}$ with eigenvalue λ_0 from Proposition 5.1.2. Statement (c) implies λ_0 being the largest non-zero eigenvalue of $\mathcal{L}$ and the existence of a spectral gap. We sketch the proofs of (b) and (c) as η and λ_0 will be crucial objects in this chapter. Note that Champagnat and Villemonais [25] are working in the more general setting of measurable spaces and therefore without the notion of a boundary. In their case, the role of the boundary is replaced by a cemetery state $\{\partial\}$. In the proof of the following theorem, we account for this slight technical difference which does not change anything about the statements made in this chapter.

Theorem 5.1.3 (Exponential convergence to QSD and dominant survival rate as eigenvalue of the generator)**.**

(a) Assumption (A) is equivalent to the existence of a unique quasi-stationary probability measure ν on E and two constants $C, \gamma > 0$ such that for all initial distributions μ on E

$$\|\mathbb{P}_\mu(X_t \in \cdot | T > t) - \nu(\cdot)\|_{TV} \leq C e^{-\gamma t} \text{ for all } t \geq 0\,, \tag{5.1.4}$$

where $\|\mathbb{P} - \mathbb{Q}\|_{TV} := \sup_{A \in \mathcal{F}} |\mathbb{P}(A) - \mathbb{Q}(A)|$ denotes the total variation distance for probability measures. In words: the process starting from any initial distribution μ, in particular $\mu = \delta_x$ for $x \in E$, converges exponentially fast to the QSD.

(b) In the situation of (a), we can define a non-negative function η on $E \cup \partial E$, positive on E and vanishing on ∂E, by

$$\eta(x) := \lim_{t\to\infty} \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)} = \lim_{t\to\infty} e^{-\lambda_0 t} \mathbb{P}_x(T > t)\,, \tag{5.1.5}$$

where the convergence holds uniformly in $E \cup \partial E$ and $\int \eta \,\mathrm{d}\nu = 1$.

Furthermore, η is a bounded eigenfunction of the infinitesimal generator $\mathcal{L}$ of the semi-group $(P_t)_{t\geq 0}$ on $(\tilde{\mathcal{E}}, \|\cdot\|_\infty)$ with eigenvalue λ_0, i.e.

$$\mathcal{L}\eta = \lambda_0 \eta\,,$$

where $-\lambda_0$ is the exponential escape rate as in Proposition 5.1.2.

(c) Let Assumption (A) hold and $f \in \tilde{\mathcal{E}}$ be an eigenfunction of $\mathcal{L}$ for an eigenvalue λ, being constant on ∂E. Then either

(i) $\lambda = 0$ and f is constant,

(ii) or $\lambda = \lambda_0$, $f = \left(\int f \,\mathrm{d}\nu\right) \eta$ and $f|_{\partial E} \equiv 0$,

(iii) or $\lambda \leq \lambda_0 - \gamma$, $\int f \,\mathrm{d}\nu = 0$ and $f|_{\partial E} \equiv 0$.

Proof. Part (a) is a shortened version of [25, Theorem 2.1].

Part (b) is contained in [25, Proposition 2.3]. Its proof uses the following fact: Let $\mathcal{M}_1(E)$ denote the set of probability measures on E. Then it can be shown that Assumption (A) implies that for any $\mu \in \mathcal{M}_1(E)$ the constant $c_2(\mu)$ defined by

$$c_2(\mu) := \inf_{t\geq 0, \rho \in \mathcal{M}_1(E)} \frac{\mathbb{P}_\mu(T > t)}{\mathbb{P}_\rho(T > t)} \tag{5.1.6}$$

is positive. This implies immediately that $\eta(x)$ is positive if it exists. Its existence follows from showing (by using (a)) that

$$\sup_{x\in E} |\eta_{t+s}(x) - \eta_t(x)| \leq \frac{C}{c_2(\nu)} e^{-\gamma t}\,,$$

where

$$\eta_t(x) := \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)}\,.$$

That η is vanishing on ∂E follows directly from its definition.

The claim in (c) is essentially [25, Corollary 2.4]. Let $\mathcal{L}f = \lambda f$. By definition of the generator, we obtain

$$\mathbb{E}_x(f(X_t)) = P_t f(x) = e^{\lambda t} f(x)\,.$$

When $f|_{\partial E} \equiv c \neq 0$, we observe, by taking $x \in \partial E$, that $\lambda = 0$. On the other hand for any $x \in E$, the left hand side converges to c and therefore f is constant. This shows (i).

Let now $f|_{\partial E} \equiv 0$. This entails together with (a) that

$$\frac{P_t f(x)}{P_t \mathbb{1}_E(x)} = \frac{\mathbb{E}_x[f(X_t)]}{\mathbb{P}_x(X_t \in E)} = \frac{\mathbb{E}_x[f(X_t)\mathbb{1}_E(X_t)]}{\mathbb{E}_x[\mathbb{1}_E(X_t)]} = \mathbb{E}_x[f(X_t)|T > t] \xrightarrow{t\to\infty} \int f\,\mathrm{d}\nu$$

uniformly in $x \in E$ and exponentially fast. To obtain (ii), we first assume that $\int f\,\mathrm{d}\nu \neq 0$. Then we obtain from (b) and the above that for all $x \in E$

$$\frac{e^{(\lambda-\lambda_0)t} f(x)}{\eta(x)} = \frac{e^{-\lambda_0 t}\mathbb{P}_x(T > t)}{\eta(x)} \frac{P_t f(x)}{P_t \mathbb{1}_E(x)} \xrightarrow{t\to\infty} \int f\,\mathrm{d}\nu\,.$$

This implies that $\lambda = \lambda_0$ and $f(x) = \left(\int f\,\mathrm{d}\nu\right)\eta(x)$ for all $x \in E$.

We finally assume $\int f\,\mathrm{d}\nu = 0$. By definition of c_2 we deduce that for all $x \in E$

$$c_2(\nu)e^{(\gamma+\lambda-\lambda_0)t} f(x) \leq \frac{e^{\gamma t} P_t f(x)}{P_t \mathbb{1}_E(x)} \xrightarrow{t\to\infty} \int f\,\mathrm{d}\nu\,.$$

The right hand-side is bounded by (a) and, hence, we obtain $\gamma + \lambda + \lambda_0 \leq 0$, which shows (iii). □

Quasi-ergodic distributions

By a classical application of the Monotone-Class Theorem [80, Theorem II.3.1], Theorem 5.1.3 implies that for all bounded and measurable functions $h : E \to \mathbb{R}$ we have

$$\lim_{t\to\infty} \mathbb{E}_x(h(X_t)|T > t) = \int_E h(y)\nu(\mathrm{d}y) \quad \text{uniformly in } x \in E\,. \tag{5.1.7}$$

However, as we want to study ergodic quantities like Lyapunov exponents, we are interested in time averages. This motivates the following definition.

Definition 5.1.4 (QED)**.** A probability measure m on E is called *quasi-ergodic distribution* (QED) if for all $t > 0$, every bounded and measurable function $h : E \to \mathbb{R}$ and every $x \in E$, the following limit exists and satisfies

$$\lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f(X_s)\mathrm{d}s \,\middle|\, T > t\right) = \int_E f\mathrm{d}m\,. \tag{5.1.8}$$

The next theorem tells us that in the situation of Theorem 5.1.3 the quasi-ergodic distribution m exists and is absolutely continuous with respect to the quasi-stationary distribution ν. The density is exactly the function η from Theorem 5.1.3. In the proof we follow He et al. [49] who made this observation very recently based on [25]. We give the whole proof here as we will use its techniques later on.

Theorem 5.1.5 (Existence of unique QED)**.** *Assume that the process $(X_t)_{t\geq 0}$ on $E \cup \partial E$ with killing at the boundary ∂E satisfies Assumption A. Then $(X_t)_{t\geq 0}$ has a unique quasi-ergodic distribution m, where the convergence in* (5.1.8) *is uniform over all $x \in E$ and m possesses a density*

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)\,.$$

Proof. Observe from (5.1.5) that $\int_E m(\mathrm{d}x) = \int_E \eta(x)\nu(\mathrm{d}x) = 1$. So m is a probability measure on E. Fix $u > 0$ and define $h_u : E \to \mathbb{R}_0^+$ by

$$h_u(x) = \inf_{t\geq u} \left(e^{-\lambda_0 t} \frac{\mathbb{P}_x(T > t)}{\eta(x)} \right) .$$

Let $f : E \to \mathbb{R}_0^+$ be bounded and measurable. Let further be $0 < q < 1$ and $(1 - q)t \geq u$. Then we obtain for all $x \in E$ that

$$\begin{aligned} \mathbb{E}_x(f(X_{qt})|T > t) &= \frac{\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>t\}}]}{\mathbb{P}_x(T > t)} \\ &= \frac{\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}\mathbb{P}_{X_{qt}}(T > (1 - q)t)]}{\mathbb{P}_x(T > t)}\,, \end{aligned}$$

where we used the Markov property. Hence, we can infer that

$$\begin{aligned} \mathbb{E}_x(f(X_{qt})|T > t) &= \frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}e^{-\lambda_0(1-q)t}\mathbb{P}_{X_{qt}}(T > (1 - q)t)]}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)} \\ &\geq \frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})\mathbb{1}_{\{T>qt\}}h_u(X_{qt})\eta(X_{qt})]}{e^{-\lambda_0 t}\mathbb{P}_x(T > t)}\,, \end{aligned}$$

According to Theorem 5.1.3 η is bounded and the convergence of the limit, via which η is defined is uniform in x. Hence, there exists a constant $C > 0$ such that for all $t \geq u$ and $x \in E$

$$|f(x)h_u(x)\eta(x)| \leq \left| f(x)e^{-\lambda_0 t}\mathbb{P}_x(T > t) \right| \leq \|f\|_\infty C\|\eta\|_\infty\,. \tag{5.1.9}$$

Thus, the function $fh_u\eta$ is bounded and obviously measurable. We observe from property

(5.1.7), the definition of η and the above that uniformly over all $x \in E$

$$\begin{aligned}\liminf_{t\to\infty} \mathbb{E}_x(f(X_{qt})|T>t) &\geq \lim_{t\to\infty} \frac{e^{-\lambda_0 qt}\mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})\mathbb{1}_{\{T>qt\}}]}{e^{-\lambda_0 t}\mathbb{P}_x(T>t)} \\ &= \lim_{t\to\infty} \frac{e^{-\lambda_0 qt}\mathbb{P}_x(T>qt)}{e^{-\lambda_0 t}\mathbb{P}_x(T>t)}\mathbb{E}_x[f(X_{qt})h_u(X_{qt})\eta(X_{qt})|T>qt] \\ &= \int_I f(x)h_u(x)\eta(x)\nu(\mathrm{d}x)\,.\end{aligned}$$

Due to (5.1.9) and the fact that $h_u(x) \to 1$ for all $x \in E$, we can apply the dominated convergence theorem to conclude that

$$\liminf_{t\to\infty} \mathbb{E}_x(f(X_{qt})|T>t) \geq \int_I f(x)m(\mathrm{d}x)\,.$$

Replacing f by $\|f\|_\infty - f$, we can see easily that

$$\limsup_{t\to\infty} \mathbb{E}_x(f(X_{qt})|T>t) \leq \int_I f(x)m(\mathrm{d}x)\,.$$

Therefore we have shown that for all bounded and positive functions f

$$\lim_{t\to\infty} \mathbb{E}_x(f(X_{qt})|T>t) = \int_I f(x)m(\mathrm{d}x)\,,$$

uniformly over $x \in E$. We can extend the result to arbitrary measurable and bounded f by writing $f = f_+ - f_-$ for $f_+ := \max\{f, 0\}$, $f_- := -\min\{f, 0\}$ and using linearity. Finally, with a change of variables, Fubini and the dominated convergence theorem we obtain for all bounded and measurable functions $f : E \to \mathbb{R}$ that uniformly over $x \in E$

$$\begin{aligned}&\lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f(X_s)\,\mathrm{d}s|T>t\right) = \lim_{t\to\infty} \mathbb{E}_x\left(\int_0^1 f(X_{qt})\,\mathrm{d}q|T>t\right) \\ &= \lim_{t\to\infty} \int_0^1 \mathbb{E}_x\left(f(X_{qt})|T>t\right)\,\mathrm{d}q = \int_E f(x)\,m(\mathrm{d}x)\,.\end{aligned}$$

Uniqueness is immediate from the definition of a quasi-ergodic distribution. □

The measure m has an additional meaning as we learn in [25]. We will see that we can define the Q-process (Y_t) with probabilities $(\mathbb{Q}_x)_{x\in E}$ such that for any $s \geq 0$

$$\mathbb{Q}_x((Y_u)_{0\leq u\leq s} \in \cdot) = \lim_{t\to\infty} \mathbb{P}_x((X_u)_{0\leq u\leq s} \in \cdot|T>t)\,.$$

The Q-process is also called the survival process since its finite-time distributions equal the ones of the original process $(X_t)_{t\geq 0}$ conditioned on asymptotic survival. We will come back to the potential role the Q-process could play within a random dynamical systems theory of killed processes.

The measure m turns out to be the unique invariant probability measure of the Markov semigroup associated with Y_t. We give the following short version of [25, Theorem 3.1]:

Theorem 5.1.6 (Q-process and QED). *Assumption (A) implies*

(i) the existence of the Q-process: there exists a family $(\mathbb{Q}_x)_{x\in E}$ *of probability measures on* Ω *defined by*

$$\lim_{t\to\infty} \mathbb{P}_x(A|T > t) = \mathbb{Q}_x(A)$$

for all $\mathcal{F}_s$*-measurable sets* A *for any given* $s \geq 0$. *The process* $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$ *on* $(\Omega, (\mathcal{F}_t)_{t\geq 0})$ *is an* E*-valued time-homogeneous Markov process. In addition, if* $(X_t)_{t\geq 0}$ *is a strong Markov process under* $(\mathbb{P}_x)_{x\in E}$*, then so is* (Y_t) *under* $(\mathbb{Q}_x)_{x\in E}$*.*

(ii) exponential ergodicity: the probability measure m *on* E *defined by*

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)$$

is the unique invariant distribution of $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$*. Furthermore, there are* $C_1, \gamma_1 > 0$ *such that for any initial distribution* μ *on* E *we obtain*

$$\|\mathbb{Q}_\mu(Y_t \in \cdot) - m(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t} \quad \text{for all } t \geq 0\,.$$

Proof. See [25, Theorem 3.1] which we have slightly reformulated for convenience of the reader. □

5.1.2 Stochastic differential equations

Consider the Markov process $(X_t)_{t\geq 0}$ as a solution of a stochastic differential equation

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + g(X_t) \circ dW_t,\ X_0 \in E\,, \tag{5.1.10}$$

in a bounded connected domain $E \subset \mathbb{R}^d$ with absorption at the boundary ∂E which is assumed to be C^2. (W_t) denotes some r-dimensional standard Brownian motion, $f : E \to \mathbb{R}^d$ a Lipschitz-continuous vector field and $g : E \to \mathbb{R}^{d\times r}$ is a differentiable, Lipschitz-continuous matrix-valued map such that gg^* is uniformly elliptic and the Itô-Stratonovich correction term is also Lipschitz continuous (see Appendix 1.4). Champagnat et al. [24] can then prove a result which immediately implies the following:

Theorem 5.1.7 (QSD and QED for stochastic differential equations). *If* $(X_t)_{t\geq 0}$ *is the solution process of the stochastic differential equation* (5.1.10) *in a bounded connected domain* $E \subset \mathbb{R}^d$ *with absorption at the* C^2 *boundary* ∂E *and* f *and* g *are as above, Assumption (A) is satisfied. In particular,*

(a) there is a QSD ν *and* $C > 0, \gamma > 0$ *such that for all probability measures* μ *on* E

$$\|\mathbb{P}_\mu(X_t \in \cdot|T > t) - \nu(\cdot)\|_{TV} \leq Ce^{-\gamma t},\ \text{for all } t \geq 0\,.$$

Furthermore, there is a subset $D \subset \mathcal{D}(\mathcal{L})$ of the domain of the generator $\mathcal{L}$ on $E \cup \partial E$ such that

$$\int \mathcal{L} f \,\mathrm{d}\nu = \lambda_0 \int f \,\mathrm{d}\nu\,, \quad \textit{for all } f \in D\,,$$

i.e. ν is an eigenmeasure of $\mathcal{L}^*$ for the largest non-zero eigenvalue $\lambda_0 < 0$ of $\mathcal{L}$ with Dirichlet boundary conditions on $E \cup \partial E$. As before, $-\lambda_0$ is the exponential escape rate as given in Proposition 5.1.2.

(b) there is a QED m given by

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)\,,$$

where

$$\eta(x) = \lim_{t\to\infty} \frac{\mathbb{P}_x(T > t)}{\mathbb{P}_\nu(T > t)} = \lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{P}_x(T > t)$$

is a bounded eigenfunction of $\mathcal{L}$ for eigenvalue λ_0, as in Theorem 5.1.3 (b).

(c) the QED m is the unique invariant distribution of the Q-process $((Y_t)_{t\geq 0}, (\mathbb{Q}_x)_{x\in E})$. Furthermore, there are $C_1, \gamma_1 > 0$ such that for any initial distribution μ on E we obtain

$$\|\mathbb{Q}_\mu(Y_t \in \cdot) - m(\cdot)\|_{TV} \leq C_1 e^{-\gamma_1 t} \quad \textit{for all } t \geq 0\,.$$

Proof. See [24, Theorem 3.1] for showing that the process satisfies Assumption (A). The statement about ν being an eigenmeasure of $\mathcal{L}^*$ is a direct consequence of [74, Proposition 4]. The other implications are taken from Theorem 5.1.3 and 5.1.5 applying the statements to the situation of the SDE (5.1.10). □

Let us now consider a case where ν and m can be determined as eigenfunctions of the generator $\mathcal{L}$. For that purpose we study a special case of (5.1.10), namely a stochastic differential equation of the form

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma\,\mathrm{d}W_t\,, \tag{5.1.11}$$

where $X_0 \in I$, $I = (l, r)$ for some $l, r \in \mathbb{R}$ and $f \in C^1(I) \cap C(\bar{I})$. First observe that if we consider the process on the real line and if $\exp\left(\int_{-\infty}^x f(y)\mathrm{d}y\right)$ is integrable, the process has a stationary measure with density

$$p(x) = \frac{1}{Z} \exp\left(\frac{2}{\sigma^2}\int_{-\infty}^x f(y)\mathrm{d}y\right)\,,$$

where $Z > 0$ is the normalisation constant. Following [28, Section 6.1.1] we define

$$\gamma(x) := \frac{2}{\sigma^2}\int_l^x f(y)\,\mathrm{d}y\,.$$

Furthermore, we define the measure

$$\mu(\mathrm{d}x) := \exp\left(\gamma(x)\right)\mathrm{d}x$$

on I. Consider the generator of the semigroup associated with (5.1.11)

$$\mathcal{L}\cdot = \frac{\sigma^2}{2}\partial_{xx}\cdot + f\partial_x\cdot\,, \tag{5.1.12}$$

with Dirichlet boundary conditions at $x = l$ and $x = r$, and its formal adjoint

$$\mathcal{L}^*\cdot = \frac{\sigma^2}{2}\partial_{xx}\cdot - \partial_x(f\cdot)\,. \tag{5.1.13}$$

With standard theory (see e.g. [27, Chapter 7]), we observe that $\mathcal{L}$ is self-adjoint in $L^2([l,r], \mathrm{d}\mu)$ and possesses a complete orthonormal basis of eigenfunctions in $L^2([l,r], \mathrm{d}\mu)$. We can apply Theorem 5.1.3 or deploy the well-known theory for one-dimensional second-order linear ODEs, as used in [28, Lemma 6.1], to observe the following properties of the eigenvalues $\lambda_n \neq 0$ for $n \geq 0$: Each λ_n is simple, negative and the only possible accumulation point of the set $\{\lambda_n : n \geq 1\}$ is $-\infty$. As before, we write without loss of generality

$$0 > \lambda_0 > \lambda_1 > \cdots > \lambda_n > \lambda_{n+1} > \ldots\,.$$

We denote by ψ_{λ_n} the unique solution to

$$\mathcal{L}\psi = \lambda_n\psi,\ \psi(l) = \psi(r) = 0,\ \int_I \psi^2 \mathrm{d}\mu = 1,\ \psi'(l) > 0\,.$$

Note that ψ_{λ_n} is smooth for all $n \geq 0$ due to the ellipticity of $\mathcal{L}$. We further observe that $\phi_{\lambda_n}(x) = \psi_{\lambda_n}(x)\exp(\gamma(x))$ satisfies

$$\mathcal{L}^*\phi = \lambda_n\phi,\ \phi(l) = \phi(r) = 0,\ \phi'(0) > 0\,.$$

Following [28], we obtain the following formula for the killed semigroup P_t defined as an operator on bounded and measurable observables $h : I \to \mathbb{R}$ by

$$P_t h(x) = \mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}]\,.$$

Note that the result is even stated for the larger space $L^2(I, \mathrm{d}\mu)$.

Proposition 5.1.8. *For all t, all $x \in I$ and all $h \in L^2(I, \mathrm{d}\mu)$,*

$$\mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}] = \sum_n e^{\lambda_n t}\left(\int_I h(y)\psi_{\lambda_n}(y)\,\mu(\mathrm{d}y)\right)\psi_{\lambda_n}(x)\,.$$

Proof. See [28, Proposition 6.2]. □

From this formula we can immediately derive the following similarly to [28, Theorem 6.4]:

Theorem 5.1.9. *The unique quasi-stationary distribution ν for the process $(X_t)_{t\geq 0}$ on I derived* *from* (5.1.11) *with absorption at the boundary* ∂I *is given by*

$$\nu(\mathrm{d}x) = \frac{\psi_{\lambda_0}(x)\mu(\mathrm{d}x)}{\int \psi_{\lambda_0}(y)\mu(\mathrm{d}y)} = \frac{\phi_{\lambda_0}(x)\,\mathrm{d}x}{\int \phi_{\lambda_0}(y)\,\mathrm{d}y}\,.$$

The QED m *can be written as*

$$m(\mathrm{d}x) = \psi^2_{\lambda_0}(x)\mu(\mathrm{d}x)$$

and we have

$$\eta(x) = \left(\int_I \psi_{\lambda_0}(y)\mu(\mathrm{d}y)\right)\psi_{\lambda_0}(x)\,.$$

The statements from Theorem 5.1.7 about exponential convergence for all initial distributions hold for ν *and* m.

Proof. Obviously all assumptions are satisfied to apply Theorem 5.1.7. Hence, we know already that there is a unique QSD. Furthermore, we obtain from Proposition 5.1.8 that for all $h \in L^2([l,r],\mathrm{d}\mu)$ and $x \in I$,

$$\lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{E}_x[h(X_t)\mathbb{1}_{\{T>t\}}] = \psi_{\lambda_0}(x)\int_I h(y)\psi_{\lambda_0}(y)\,\mu(\mathrm{d}y)\,.$$

In particular, by taking $h \equiv 1$, we obtain

$$\lim_{t\to\infty} e^{-\lambda_0 t}\mathbb{P}_x(T>t) = \psi_{\lambda_0}(x)\int_I \psi_{\lambda_0}y)\,\mu(\mathrm{d}y)\,.$$

So we can deduce that for all bounded and measurable functions h and $x \in I$ we have the limit

$$\lim_{t\to\infty} \mathbb{E}_x(h(X_t)|T>t) = \int_I h(y)\,\nu(\mathrm{d}y)\,,$$

where ν is given as in the statement of the theorem. Hence, ν is the unique QSD. From the fact that η has to be proportional to ψ_{λ_0} and the normalisation condition on ψ_{λ_0}, we get the expressions for m and η. □

5.2 Lyapunov exponents and local stability

We are turning to the study of killed processes from a random dynamical systems perspective. In this section, we mainly investigate the existence of asymptotic average Lyapunov exponents and the convergence behaviour of finite-time exponents to such quantities.

Let $(X_t)_{t\geq 0}$ be a time-homogeneous Markov process on a topological state space E with absorption at the boundary ∂E, where the process possesses the family of transition probabilities $(\mathbb{P}_x)_{x\in E}$ on a filtered probability space $(\Omega,\mathcal{F},(\mathcal{F}_t)_{t\geq 0},\mathbb{P})$. Further, let there be a random dynamical system $(\theta,\hat{\varphi})$ associated with this process such that

$$\mathbb{P}_x(X_t \in B) = \mathbb{P}(\hat{\varphi}(t,\cdot,x) \in B) \quad \text{for all} \;\; t \geq 0, x \in E, B \in \mathcal{B}(\bar{E})\,.$$

We encounter such a setting, for example, if the random dynamical system is induced by (5.1.10) in the canonical way. In such a situation, we can make the following definition.

Definition and Lemma 5.2.1. Let $(\theta, \hat{\varphi})$ denote the random dynamical system associated with a Markov process $(X_t)_{t\geq 0}$ on a topological state space E with absorption at the boundary ∂E. Define $\tilde{T} : \Omega \times E \to \mathbb{R}_0^+$ as

$$\tilde{T}(\omega, x) = \inf\{t > 0 \,:\, \hat{\varphi}(t, \omega, x) \in \partial E\}$$

such that for all $x \in E$ and $t \geq 0$

$$\mathbb{P}_x(T > t) = \mathbb{P}(\tilde{T}(\cdot, x) > t)\,.$$

Then together with (θ, Ω), the map $\varphi : \mathbb{R}_0^+ \times \Omega \times \bar{E} \to \bar{E}$ given by

$$\varphi(t, \omega, x) = \begin{cases} \hat{\varphi}(t, \omega, x) & \text{if } t < \tilde{T}(\omega, x)\,, \\ \hat{\varphi}(\tilde{T}(\omega, x), \omega, x) & \text{if } t \geq \tilde{T}(\omega, x)\,, \\ x & \text{if } x \in \partial E\,. \end{cases}$$

constitutes a random dynamical system.

Proof. Measurability and $\varphi(0, \omega, \cdot) = \mathrm{id}$ are clear from the definition. It remains to check the cocycle property by distinguishing different cases. Fix $(\omega, x) \in \Omega \times \bar{E}$. First note from the cocycle property of the original system $\hat{\varphi}$ that for $s < \tilde{T}(\omega, x)$ we have

$$\hat{\varphi}(\tilde{T}(\omega, x) - s, \theta_s\omega, \hat{\varphi}(s, \omega, x)) = \hat{\varphi}(\tilde{T}(\omega, x), \omega, x)\,,$$

and therefore

$$\tilde{T}(\theta_s\omega, \varphi(s, \omega, x)) = \tilde{T}(\theta_s\omega, \hat{\varphi}(s, \omega, x)) = \tilde{T}(\omega, x) - s\,. \tag{5.2.1}$$

Hence, if $t + s < \tilde{T}(\omega, x)$, it follows that

$$\varphi(t + s, \omega, x) = \hat{\varphi}(t + s, \omega, x) = \hat{\varphi}(t, \theta_s\omega, \hat{\varphi}(s, \omega, x)) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))\,.$$

Now let $t + s \geq \tilde{T}(\omega, x)$: If $t, s \geq \tilde{T}(\omega, x)$, we have by definition of φ

$$\varphi(t + s, \omega, x) = \hat{\varphi}(\tilde{T}(\omega, x), \omega, x) = \varphi(t, \theta_s\omega, \hat{\varphi}(\tilde{T}(\omega, x), \omega, x)) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))\,.$$

If w.l.o.g $s < \tilde{T}(\omega, x)$, we obtain from (5.2.1) that $t \geq \tilde{T}(\omega, x) - s \geq \tilde{T}(\theta_s\omega, \varphi(s, \omega, x))$ and therefore

$$\varphi(t + s, \omega, x) = \varphi(\tilde{T}(\omega, x), \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))\,.$$

This concludes the proof. □

Remark 5.2.2. Note that Definition and Lemma 5.2.1 defines a local random dynamical system in the sense of [2, Definition 1.2.1] with extension to the boundary. The domain $D \subset \mathbb{R}_0^+ \times \Omega \times E$ of the local random dynamical system satisfies

$$D(\omega, x) = \{t \in \mathbb{R}_0^+ \, : \, (t, \omega, x) \in D\} = [0, \tilde{T}(\omega, x))\,,$$

and

$$D(t, \omega) = \{x \in E \, : \, (t, \omega, x) \in D\} = \{x \in E \, : \, \tilde{T}(\omega, x) > t\}\,.$$

The classical motivation to consider local random dynamical systems is the possible explosion of solutions for a stochastic or random differential equation in an unbounded state space outside the domain D. If the eternal survival sets

$$E(\omega) = \bigcap_{t \in \mathbb{R}_0^+} D(t, \omega)$$

are non-empty almost surely, any invariant random measure has to be supported on these fibres and the formulation of a Multiplicative Ergodic Theorem is possible for such an invariant measure. However, in the situations we are interested in, i.e. stochastic differential equations with additive noise on a bounded domain $E \subset \mathbb{R}^d$, $E(\omega)$ is empty almost surely. Hence, the problem demands for a new method describing asymptotic expansion and contraction rates, using the idea of quasi-ergodic distributions.

Note that φ, as defined in Definition and Lemma 5.2.1, is not continuous at $x \in \partial E$ for all $t > 0$ and $\omega \in \Omega$, so φ isn't a continuous random dynamical system in the sense of Definition 1.1.1. However, for any x in E and $\omega \in \Omega$, the system φ is continuous in x for all $t < \tilde{T}(\omega, x)$, and even differentiable if $\hat{\varphi}$ is. In the situation of (5.1.10) this is the case if the coefficients are differentiable. We can consider the finite-time Lyapunov exponents

$$\lambda_v(t, \omega, x) = \frac{1}{t} \ln \frac{\|\mathrm{D}\varphi(t, \cdot, x)v\|}{\|v\|} \quad \text{for } t < \tilde{T}(\omega, x)\,,$$

where $\mathrm{D}\varphi = D\hat{\varphi}$ solves the variational equation corresponding to (5.1.10) given by

$$\mathrm{d}Y(t, \omega, x) = \mathrm{D}f(\varphi(t, \omega, x))Y(t, \omega, x)\,\mathrm{d}t + \sum_{j=1}^{r} Dg^j(\varphi(t, \omega, x))Y(t, \omega, x) \circ \mathrm{d}W_t^j\,, \tag{5.2.2}$$

where $Y(0, \omega, x) = \mathrm{Id}$, and g^j denotes the j-th column of g and W_t^j the j-th entry of W_t. We want to investigate the convergence behaviour of the finite-time Lyapunov exponents under conditioning to absorption at the boundary.

We restrict ourselves to problems with additive noise as this will be enough for our most relevant examples. Consider the stochastic differential equation

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sigma\,\mathrm{d}W_t\,,\ X_0 = \xi \in E\,, \tag{5.2.3}$$

where $E \subset \mathbb{R}^d$ is a bounded connected domain with C^2 boundary ∂E and $f : E \to \mathbb{R}^d$ is a continuously differentiable vector field with bounded derivative. Then all the conditions of Theorem 5.1.7 are satisfied. Hence, the quasi-stationary distribution ν on E is a limiting distribution whose density ϕ vanishes at the boundary and satisfies

$$\mathcal{L}^*\phi = \lambda_0\phi\,,$$

where $\mathcal{L}^*$ is the formal adjoint of the generator $\mathcal{L}$ which is given by

$$\mathcal{L} = f\cdot\nabla + \frac{1}{2}\sigma^2\nabla^2\,.$$

Furthermore we know that

$$\mathcal{L}\eta = \lambda_0\eta\,,$$

and that the quasi-ergodic distribution m satisfies

$$m(\mathrm{d}x) = \eta(x)\nu(\mathrm{d}x)\,.$$

Hence, in principle these measures can be calculated explicitly.

The Jacobian $\mathrm{D}\varphi$ of the RDS is the solution of the variational equation, which in this case reads

$$\frac{\mathrm{d}}{\mathrm{d}t}Y(t,\omega,x) = \mathrm{D}f(\varphi(t,\omega,x))Y(t,\omega,x), \quad Y(0,\omega,x) = \mathrm{Id}\,. \tag{5.2.4}$$

A first question to ask is if there are limits for the average finite-time Lyapunov exponents as in the classical case. That means that we want to find out if for $v\in\mathbb{R}^d\setminus\{0\}$ there are $\lambda_v\in\mathbb{R}$ such that for all $x\in E$

$$\lambda_v := \lim_{t\to\infty}\mathbb{E}\left[\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right] = \lim_{t\to\infty}\frac{1}{t}\mathbb{E}\left[\ln\frac{\|\mathrm{D}\varphi(t,\cdot,x)v\|}{\|v\|}\middle|\tilde{T}(\cdot,x)>t\right]\,. \tag{5.2.5}$$

Indeed, we can show the following modified Furstenberg-Khasminskii-formula:

Proposition 5.2.3 (Conditioned average Lyapunov exponent)**.** *Let (θ,φ) be the random dynamical system with absorption at the boundary corresponding to the Markov process $(X_t)_{t\geq 0}$ solving equation* (5.2.3)*, and let*

$$s_t = \frac{\mathrm{D}\varphi(t,\cdot,\cdot)}{\|\mathrm{D}\varphi(t,\cdot,\cdot)\|}$$

denote the induced process on the unit sphere of the tangent space. If the generator $\tilde{\mathcal{L}}$ of $(X_t,s_t)_{t\geq 0}$ is hypoelliptic, then for all $v\in\mathbb{R}^d\setminus\{0\}$ the average exponent λ_v as defined in (5.2.5) *exists and is given independently from v by*

$$\lambda_v = \lambda := \int_{\mathbb{S}^{d-1}\times E}\langle s,\mathrm{D}f(y)s\rangle\;\tilde{m}(\mathrm{d}s,\mathrm{d}y), \tag{5.2.6}$$

where $\tilde{m}$ is the quasi-ergodic joint distribution of $(X_t,s_t)_{t\geq 0}$ and the convergence is uniform over all $x\in E$ and $v\in\mathbb{R}^d\setminus\{0\}$. We call λ the conditioned average Lyapunov exponent.

Proof. Note that the angular component s_t as defined above lies on the unit sphere $\mathbb{S}^{d-1}$ and write $r_t = \|\mathrm{D}\varphi(t,\cdot,\cdot)\|$ for the radial component. The variational equation (5.2.4) in vector polar coordinates is given by

$$\begin{aligned} \mathrm{d}s_t &= \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t - \langle s_t, \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t\rangle s_t \,\mathrm{d}t\,,\ s_0 \in \mathbb{S}^{d-1}\,,\\ \mathrm{d}r_t &= \langle s_t, \mathrm{D}f(\varphi(t,\cdot,\cdot))s_t\rangle r_t \,\mathrm{d}t\,,\ r_0 = 1\,. \end{aligned}$$

We obtain for all $\omega \in \Omega$, $x \in E$

$$r_t(\omega,x) = r_0 \exp\left(\int_0^t h(\varphi(\tau,\omega,x), s_\tau(\omega,x))\,\mathrm{d}\tau\right)\,,$$

where $h : E \times \mathbb{S}^{d-1} \to \mathbb{R}$ is given by

$$h(x,s) = \langle s, \mathrm{D}f(x)s\rangle\,.$$

We observe that $(X_t, s_t)_{t\geq 0}$ constitutes a skew product system on $E \times \mathbb{S}^{d-1}$ with killing at $\partial E \times \mathbb{S}^{d-1}$. To apply the theory of Section 5.1.1, we need to check that Assumption (A) is satisfied. We know that $(X_t)_{t\geq 0}$ satisfies the assumption on $E \cup \partial E$, i.e. there is a family (ν_{x_1,x_2}) that fulfils (A1) and (A2) for some constants $t_0, c_1, c_2 > 0$. Due to the hypoellipticity condition, there exists a $c_0 > 0$ and a family of probability measures (μ_{z_1,z_2}) such that for any $z_i = (x_i, s_i) \in E \times \mathbb{S}^{d-1}, i = 1, 2$ and $A \in \mathcal{B}(E)$ with $\nu_{x_1,x_2}(A) > 0$

$$\mathbb{P}_{z_i}(s_{t_0} \in \cdot | X_{t_0} \in A, T > t_0) \geq c_0 \mu_{z_1,z_2}(\cdot)\,,$$

for similar reasons as in the proof of [24, Theorem 3.1]. We define the family of probability measures

$$\tilde{\nu}_{z_1,z_2}(A \times B) = \nu_{x_1,x_2}(A)\mu_{z_1,z_2}(B) \ \text{ for all measurable } A \subset E, B \subset \mathbb{S}^{d-1}, z_1, z_2 \in E \times \mathbb{S}^{d-1}\,.$$

Since $(X_t)_{t\geq 0}$ and therefore T are independent from $(s_t)_{t\geq 0}$, we observe that for all $z_1, z_2 \in E \times \mathbb{S}^{d-1}$ and measurable $A \subset E, B \subset \mathbb{S}^{d-1}$

$$\begin{aligned} \mathbb{P}_{z_i}((X_{t_0}, s_{t_0}) \in A \times B | T > t_0) &= \frac{\mathbb{P}_{z_i}((X_{t_0}, s_{t_0}) \in A \times B, T > t_0)}{\mathbb{P}_{x_i}(T > t_0)}\\ &= \mathbb{P}_{z_i}(s_{t_0} \in B | X_{t_0} \in A, T > t_0)\mathbb{P}_{x_i}(X_{t_0} \in A | T > t_0)\\ &\geq c_0\mu_{z_1,z_2}(B)\mathbb{P}_{x_i}(X_{t_0} \in A | T > t_0) \geq c_0 c_1 \tilde{\nu}_{z_1,z_2}(A \times B)\,. \end{aligned}$$

This shows (A1). Using again the independence of the hitting time T from s_t, (A2) follows by observing that for all $z_1, z_2 \in E \times \mathbb{S}^{d-1}$

$$\mathbb{P}_{\nu_{z_1,z_2}}(T > t) = \int_{\mathbb{S}^{d-1}} \int_E \mathbb{P}_x(T > t)\,\nu_{x_1,x_2}(\mathrm{d}x)\,\mu_{z_1,z_2}(\mathrm{d}s) \geq c_2 \sup_{x \in E, s \in \mathbb{S}^{d-1}} \mathbb{P}_{x,s}(T > t)\,.$$

Theorem 5.1.3 and Theorem 5.1.5 allow us to conclude that there are a unique QSD $\tilde{\nu}$ and associated QED $\tilde{m}$ on $E \times \mathbb{S}^{d-1}$ which due to the skew product structure have ν and m as their marginals on E. Hence, by definition of a quasi-ergodic measure and the fact that h is bounded and measurable by the assumptions, we conclude that for all $v \in \mathbb{R}^d$ with $v = \|1\|$ (which is enough for the claim) and $x \in E$

$$\begin{aligned}\lambda_v &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_{x,s_0}[\ln r_t | T > t] = \lim_{t\to\infty} \frac{1}{t}\mathbb{E}_x[\ln r_t | T > t] \\ &= \lim_{t\to\infty} \frac{1}{t}\mathbb{E}\left[\int_0^t h(\varphi(\tau,\cdot,x), s_\tau(\cdot,x))\,\mathrm{d}\tau \middle| \tilde{T}(\cdot,x) > t\right] = \int h\,\mathrm{d}\tilde{m}\,,\end{aligned}$$

where the convergence is uniform according to Theorem 5.1.5. This concludes the proof of the proposition. □

Remark 5.2.4. In principle, we can extend this result to the general situation of (5.1.10). We refrain from doing this here for two reasons. Firstly, in case of a nonlinear diffusion term g, the functional that has to be added to $\langle s, \mathrm{D}f(x)s\rangle$ assumes a complicated shape including second derivatives, if the original and linearised process have interfering noise terms. We avoid the loss of clarity and comprehensibility of this situation. Secondly, the relevant examples of this thesis have additive noise terms such that the formulation of Proposition 5.2.3 suffices for our purposes.

In one dimension, the problem is reduced to considering systems on an interval $I \subset \mathbb{R}$ induced by the one-dimensional SDE (5.1.11), where $f \in C^1(I) \cap C(\bar{I})$ and f' is bounded on I. In this case the finite-time exponents are simply given by

$$\lambda(t,\omega,x) = \frac{1}{t}\ln|\mathrm{D}\varphi(t,\omega,x)| \quad \text{for } t < \tilde{T}(\omega,x)\,,$$

where $\mathrm{D}\varphi(t,\omega,x)$ solves the linear variational equation for (5.1.11)

$$\dot{v}(t,\omega,x) = f'(\varphi(t,\omega,x))v(t,\omega,x), \quad v(0,\omega,x) = 1\,, \quad \text{for all } x \in I, \omega \in \Omega.$$

So in this one-dimensional scenario we can immediately infer that

$$\lambda(t,\omega,x) = \frac{1}{t}\int_0^t f'(\varphi(s,\omega,x))\,\mathrm{d}s \;\; \text{for } t < \tilde{T}(\omega,x)\,.$$

Let us write $\psi = \psi_{\lambda_0}$ from now on. We can show the following formula for the conditioned average Lyapunov exponent λ in the one-dimensional scenario in congruence with Proposition 5.2.3.

Proposition 5.2.5 (Conditioned average Lyapunov exponent in one dimension)**.** *Let* (θ,φ) *be the random dynamical system on* $\bar{I} \subset \mathbb{R}$ *induced by* (5.1.11) *with absorption at the boundary, where* $f \in C^1(I) \cap C(\bar{I})$ *and* f' *is bounded on* I*. Then the conditioned average Lyapunov exponent*

λ is given, independently from $x \in I$, by

$$\lambda = \lim_{t\to\infty} \mathbb{E}\left(\lambda(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right) = \int_I f'(y)\, m(\mathrm{d}y) = \int_I f'(y)\psi^2(y)e^{\gamma(y)}\,\mathrm{d}y \tag{5.2.7}$$

for all $x \in I$.

Proof. The claim follows immediately from Theorem 5.1.5 by using that f' is bounded and measurable on I which implies, by definition of the QED m, that

$$\begin{aligned}\lim_{t\to\infty} \mathbb{E}\left(\lambda(t,\cdot,x)|\tilde{T}(\cdot,x) > t\right) &= \lim_{t\to\infty} \mathbb{E}\left(\frac{1}{t}\int_0^t f'(\varphi(s,\cdot,x))\,\mathrm{d}s|\tilde{T}(\cdot,x) > t\right)\\ &= \lim_{t\to\infty} \mathbb{E}_x\left(\frac{1}{t}\int_0^t f'(X_s)\,\mathrm{d}s|T > t\right) = \int_I f'(y)\, m(\mathrm{d}y)\,.\end{aligned}$$

The formula for this integral is taken from Theorem 5.1.9. □

Remark 5.2.6. To obtain a priori estimates on the sign of λ, we could try to use the fact that ψ is an eigenfunction of $\mathcal{L}$ for the eigenvalue λ_0, i.e.

$$\frac{1}{2}\sigma^2\psi''(x) + f(x)\psi'(x) = \lambda_0\psi(x)\,. \tag{5.2.8}$$

Using integration by parts, we observe that

$$\begin{aligned}\int_I f'(x)\psi^2(x)e^{\gamma(x)}\mathrm{d}x &= -\int_I f(x)\left(\frac{2}{\sigma^2}f(x)\psi^2(x)e^{\gamma(x)} + 2\psi(x)\psi'(x)e^{\gamma(x)}\right)\mathrm{d}x\\ &= \underbrace{-\frac{2}{\sigma^2}\int_I f^2(x)m(\mathrm{d}x)}_{\leq 0} - \underbrace{2\int_I f(x)\psi'(x)\psi(x)e^{\gamma(x)}\,\mathrm{d}x}_{\text{sign unclear}}\,.\end{aligned}$$

Note that the second term vanishes in case $I = \mathbb{R}$ since ψ is a constant function in this case and m is the stationary distribution. That is how one observes negativity of the Lyapunov exponent in the classical setting. In our context, the sign of the second term depends on the product $f(x)\psi'(x)$ which makes a direct a priori estimate impossible. Using (5.2.8), we can rewrite it as

$$\begin{aligned}\int_I f'(x)\psi^2(x)e^{\gamma(x)}\mathrm{d}x &= -\int_I f(x)\left(\frac{2}{\sigma^2}f(x)\psi^2(x)e^{\gamma(x)} + 2\psi(x)\psi'(x)e^{\gamma(x)}\right)\mathrm{d}x\\ &= \underbrace{-\frac{2}{\sigma^2}\int_I f^2(x)\,m(\mathrm{d}x)}_{\leq 0}\underbrace{-2\lambda_0}_{>0} + \underbrace{\sigma^2\int_I \psi''(x)\psi(x)e^{\gamma(x)}\,\mathrm{d}x}_{\text{sign unclear}}\,.\end{aligned}$$

Again, we remain with a term whose sign is unclear, this time depending on $\psi''(x)$. It appears not possible to obtain general statements about the sign of λ.

We observe that λ as defined in Proposition 5.2.3 (and given in Proposition 5.2.5 for one dimension) is defined as a limit of conditioned expected values. In random dynamical system theory, however, we are usually interested in ω-wise asymptotic statements. Due to the killing at the boundary, the best we can hope for in this context is a convergence result of finite-time Lyapunov exponents in probability. Indeed, we are able to prove such a result in Theorem 5.2.8 which shows that this number λ actually has a dynamical meaning. Before we can prove the Theorem, we show the following Lemma about decay of correlations:

Lemma 5.2.7. *Let $(X_t)_{t\geq 0}$ be a Markov process on a topological state space E with absorption at the boundary ∂E as introduced in Section 5.1.1. Then for any measurable and bounded functions $f, g : E \to \mathbb{R}$, $0 < r < q < 1$ we have*

$$\lim_{t\to\infty} \mathbb{E}_x \left(f(X_{qt})g(X_{rt}) | T > t \right) = \left(\int f \mathrm{d}m \right) \left(\int g \mathrm{d}m \right)$$

uniformly over all $x \in E$.

Proof. Let $f, g : X \to \mathbb{R}_0^+$ be measurable and bounded functions, $0 < r < q < 1$ and $x \in E$. Similarly to the proof of Theorem 5.1.5, we fix $u > 0$ and define the observable

$$h_u(x) = \inf\{e^{-\lambda_1 t}\mathbb{P}_x(T > t)/\eta(x) \,:\, t \geq u\}\,.$$

Let t be large enough such that $(q - r)t \geq u$ and $(1 - q)t \geq u$. We obtain with the Markov property

$$\begin{aligned}
\mathbb{E}_x \left[f(X_{qt})g(X_{rt}) | T > t \right] &= \frac{\mathbb{E}_x[g(X_{rt})f(X_{qt})\mathbb{1}_{\{T>t\}}]}{\mathbb{P}_x(T > t)} \\
&= \frac{\mathbb{E}_x[g(X_{rt})f(X_{qt})1_{\{T>qt\}}\mathbb{P}_{X_{qt}}(T > (1 - q)t)]}{\mathbb{P}_x(T > t)} \\
&\geq \frac{\mathbb{E}_x[g(X_{rt})\mathbb{1}_{\{T>qt\}}e^{-\lambda_0(q-1)t}f(X_{qt})h_u(X_{qt})\eta(X_{qt})]}{\mathbb{P}_x(T > t)}\,.
\end{aligned}$$

Let us denote $\rho(x) = f(x)h_u(x)\eta(x)$ and $\tilde{\rho}(x) = g(x)h_u(x)\eta(x)$ which are positive and bounded for the same reasons as in the proof of Theorem 5.1.5. Analogously to the above, we obtain

$$\begin{aligned}
&\mathbb{E}_x \left(f(X_{qt})g(X_{rt}) | T > t \right) \geq e^{-\lambda_0(q-1)t}\frac{\mathbb{E}_x\left[g(X_{rt})\mathbb{1}_{\{T>rt\}}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]\right]}{\mathbb{P}_x(T > t)} \\
&\geq e^{-\lambda_0(q-1)t}\mathbb{E}_x\left[\frac{g(X_{rt})h_u(X_{rt})\eta(X_{rt})\mathbb{1}_{\{T>rt\}}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_x(T > t)\mathbb{P}_{X_{rt}}(T > t)e^{-\lambda_0 t}}\right] \\
&= \mathbb{E}_x\left[\frac{e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}}{\mathbb{P}_x(T > t)e^{-\lambda_0 t}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T > t)e^{-\lambda_0 t}}\right]\,.
\end{aligned}$$

By Theorem 5.1.7 the limits of the killed semigroup are uniform in x. For the second term in the expectation above we obtain

$$\begin{aligned}
&\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T>t)e^{-\lambda_0 t}}-\int\rho\,\mathrm{d}\nu\right| \\
=&\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_{X_{rt}}(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_{X_{rt}}(T>t)}\mathbb{E}_{X_{rt}}[\rho(X_{(q-r)t})|T>(q-r)t]-\int\rho\,\mathrm{d}\nu\right| \\
\leq& Ce^{-\gamma t}+\|\rho\|_\infty\left|\mathbb{1}_{\{T>rt\}}\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_{X_{rt}}(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_{X_{rt}}(T>t)}-1\right| \\
\leq& Ce^{-\gamma t}+\|\rho\|_\infty\sup_{x\in E}\left|\frac{e^{-\lambda_0(q-r)t}\mathbb{P}_{x}(T>(q-r)t)}{e^{-\lambda_0 t}\mathbb{P}_{x}(T>t)}-\frac{\eta(x)}{\eta(x)}\right|
\end{aligned}$$

where the second term converges to 0 as $t\to\infty$ according to Theorem 5.1.7. Hence, the second factor in the expectation converges uniformly to its limit

$$\int\rho\,\mathrm{d}\nu=\int fh_u\,\mathrm{d}m\,.$$

Therefore we observe that

$$\begin{aligned}
&\liminf_{t\to\infty}\mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right) \\
&\geq\lim_{t\to\infty}\mathbb{E}_x\left[\frac{e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}}{\mathbb{P}_x(T>t)e^{-\lambda_0 t}}\frac{e^{-\lambda_0(q-r)t}\mathbb{E}_{X_{rt}}[\mathbb{1}_{\{T>(q-r)t\}}\rho(X_{(q-r)t})]}{\mathbb{P}_{X_{rt}}(T>t)e^{-\lambda_0 t}}\right] \\
&=\lim_{t\to\infty}\frac{\mathbb{E}_x\left[e^{-\lambda_0 rt}\tilde{\rho}(X_{rt})\mathbb{1}_{\{T>rt\}}\right]}{\mathbb{P}_x(T>t)e^{-\lambda_0 t}}\int fh_u\,\mathrm{d}m \\
&=\int gh_u\,\mathrm{d}m\int fh_u\,\mathrm{d}m\,.
\end{aligned}$$

Since h_u is uniformly bounded as seen in the proof of Theorem 5.1.5 and $h_u(x)\to 1$, we have by the dominated convergence theorem that

$$\liminf_{t\to\infty}\mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right)\geq\left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).$$

Replacing $f(X_{qt})g(X_{rt})$ by

$$(\|f\|_\infty-f(X_{qt}))(\|g\|_\infty+g(X_{rt}))\quad\text{and}\quad(\|f\|_\infty+f(X_{qt}))(\|g\|_\infty-g(X_{rt}))\,,$$

we can see directly that

$$\begin{aligned}
&2\|f\|_\infty\|g\|_\infty - \limsup_{t\to\infty} \mathbb{E}_x\left(2f(X_{qt})g(X_{rt}))|T>t\right) \\
&\geq \liminf_{t\to\infty} \mathbb{E}_x\left((\|f\|_\infty - f(X_{qt}))(\|g\|_\infty + g(X_{rt}))\Big|T>t\right) \\
&+ \liminf_{t\to\infty} \mathbb{E}_x\left((\|f\|_\infty + f(X_{qt}))(\|g\|_\infty - g(X_{rt}))\Big|T>t\right) \\
&\geq 2\|f\|_\infty\|g\|_\infty - \left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).
\end{aligned}$$

Therefore, we deduce that

$$\limsup_{t\to\infty} \mathbb{E}_x\left(f(X_{qt})g(X_{rt})|T>t\right) \leq \left(\int f\mathrm{d}m\right)\left(\int g\mathrm{d}m\right).$$

So, we have shown the claim for positive measurable and bounded functions f, g. We can extend the result to arbitrary measurable and bounded f, g analogously to the proof of Theorem 5.1.5 by replacing fg with $(f_+ - f_-)(g_+ - g_-)$. Uniformity of the convergence follows for the same reasons as in Theorem 5.1.5. □

We are ready to prove the following theorem which equips the limit of expected values λ, as given in Propositions 5.2.3 and 5.2.5, with the strongest possible dynamical meaning for the setting with killing at the boundary.

Theorem 5.2.8 (Convergence in conditional probability)**.** *Let (θ, φ) be the random dynamical system with absorption at the boundary corresponding to the Markov process $(X_t)_{t\geq 0}$ solving equation* (5.2.3)*. Then for all $\varepsilon > 0$ we have*

$$\lim_{t\to\infty} \mathbb{P}\left(|\lambda_v(t,\cdot,x) - \lambda| \geq \varepsilon \Big| \tilde{T}(\cdot,x) > t\right) = 0 \tag{5.2.9}$$

uniformly over all $x \in E$, $v \in \mathbb{S}^{d-1}$. This means that the finite-time Lyapunov exponents of the surviving trajectories converge to its assemble average in probability. Note that in one dimension this reads

$$\lim_{t\to\infty} \mathbb{P}\left(|\lambda(t,\cdot,x) - \lambda| \geq \varepsilon \Big| \tilde{T}(\cdot,x) > t\right) = 0\,. \tag{5.2.10}$$

Proof. Recall from above that

$$\begin{aligned}
\lambda &= \lim_{t\to\infty} \mathbb{E}_{x,s_0}\left(\lambda_v(t,\cdot,x)|T>t\right) = \lim_{t\to\infty} \mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right) \\
&= \int_{\mathbb{S}^{d-1}\times E} \langle s, \mathrm{D}f(x)s\rangle\, \tilde{m}(\mathrm{d}s,\mathrm{d}x)\,.
\end{aligned}$$

In the following, we will write $g(x,s) = \langle s, \mathrm{D}f(x)s\rangle$ and $\tilde{m}(h) := \int h\,\mathrm{d}\tilde{m}$ for any bounded and measurable function h. Note that in one dimension we have $g(x,s) = f'(x)$ and $\tilde{m} = m$. We observe that

$$\begin{aligned}
&\mathbb{P}\left(|\lambda_v(t,\cdot,x)-\lambda| \geq \varepsilon \middle| \tilde{T}(\cdot,x)>t\right) \\
&\leq \mathbb{P}\left(\left|\lambda_v(t,\cdot,x)-\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x)>t\right) \\
&+\mathbb{P}\left(\left|\lambda-\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x)>t\right).
\end{aligned}$$

The term in the second line converges to zero for t to infinity by definition of λ. The first term can be estimated by Chebyshev's inequality:

$$\mathbb{P}\left(\left|\lambda_v(t,\cdot,x)-\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right)\right| \geq \varepsilon \middle| \tilde{T}(\cdot,x)>t\right) \leq \frac{\operatorname{Var}(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t)}{\varepsilon^2}.$$

This means that, in order to prove the claim, we simply need to show that

$$\lim_{t\to\infty}\mathbb{E}\left(\lambda_v(t,\cdot,x)^2|\tilde{T}(\cdot,x)>t\right)=\lim_{t\to\infty}\left[\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right)\right]^2,$$

where

$$\lim_{t\to\infty}\left[\mathbb{E}\left(\lambda_v(t,\cdot,x)|\tilde{T}(\cdot,x)>t\right)\right]^2=\lambda^2=\tilde{m}(g)^2.$$

Similarly to the proof of Theorem 5.1.5 we obtain with Fubini that

$$\begin{aligned}
\lim_{t\to\infty}\mathbb{E}\left(\lambda_v(t,\cdot,x)^2|\tilde{T}(\cdot,x)>t\right) &= \lim_{t\to\infty}\mathbb{E}_x\left(\left(\frac{1}{t}\int_0^t g(X_\tau,s_\tau)\,\mathrm{d}\tau\right)^2|T>t\right) \\
&= \lim_{t\to\infty}\mathbb{E}_x\left(\left(\int_0^1 g(X_{qt},s_{qt})\,\mathrm{d}q\right)^2|T>t\right) \\
&= \lim_{t\to\infty}\int_0^1\int_0^1 \mathbb{E}_x\left(g(X_{qt},s_{qt})g(X_{rt},s_{rt})|T>t\right)\mathrm{d}q\,\mathrm{d}r \\
&= \lim_{t\to\infty}\left[\int_0^1\int_0^q \mathbb{E}_x\left(g(X_{qt},s_{qt})g(X_{rt},s_{rt})|T>t\right)\mathrm{d}r\,\mathrm{d}q\right. \\
&\left.+\int_0^1\int_0^r \mathbb{E}_x\left(g(X_{qt},s_{qt})g(X_{rt},s_{rt})|T>t\right)\mathrm{d}q\,\mathrm{d}r\right].
\end{aligned}$$

It follows immediately from Lemma 5.2.7 that for $0<r<q<1$ (and $0<q<r<1$)

$$\lim_{t\to\infty}\mathbb{E}_x\left(g(X_{qt},s_{qt})g(X_{rt},s_{rt})|T>t\right)=\lim_{t\to\infty}\mathbb{E}_{x,s_0}\left(g(X_{qt},s_{qt})g(X_{rt},s_{rt})|T>t\right)=\tilde{m}(g)^2,$$

where the convergence is uniform over the initial values. Hence, by using dominated convergence, we conclude that

$$\lim_{t\to\infty}\mathbb{E}\left(\lambda_v(t,\cdot,x)^2|\tilde{T}(\cdot,x)>t\right)=\int_0^1\int_0^q \tilde{m}(g)^2\,\mathrm{d}r\,\mathrm{d}q+\int_0^1\int_0^r \tilde{m}(g)^2\,\mathrm{d}q\,\mathrm{d}r=\tilde{m}(g)^2,$$

such that the claim follows. □

Proposition 5.2.3 tells us, not surprisingly, that if we take the limit of the expectation of the tangent flow, the initial vector v on the tangent space does not matter. Figuratively speaking, we average out the geometry of the dynamics. In the classical setting, this geometry is reflected by a spectrum of Lyapunov exponents associated with a filtration or splitting of flow-invariant subspaces. Something like that is not directly obtainable in our setting since such filtrations or subspaces depend on the noise realisation ω which only has a finite survival time in our context. However, we can try to find a spectrum of Lyapunov exponents following the Furstenberg-Kesten Theorem [2, Theorem 3.3.3].

For a time $t > 0$, consider $\Phi(t, \omega, x) := \mathrm{D}\varphi(t, \omega, x) \in \mathbb{R}^{d \times d}$ for all (ω, x) such that $\varphi(s, \omega, x) \in E$ for all $0 \leq s \leq t$. Let

$$0 < \sigma_d(\Phi(t, \omega, x)) \leq \cdots \leq \sigma_1(\Phi(t, \omega, x))$$

be the singular values of $\Phi(t, \omega, x)$, i.e. the eigenvalues of $\sqrt{\Phi^*(t, \omega, x)\Phi(t, \omega, x)}$. We would like to show the following:

Conjecture 5.2.9 (Lyapunov spectrum)**.** *Let Φ be the linearised flow associated with problem* (5.2.3)*. Then there are $\sigma^i \in \mathbb{R}$ such that for all $x \in E$ we have*

$$\frac{1}{t}\mathbb{E}[\ln \sigma_i(\Phi(t, \cdot, x)) | \tilde{T}(\cdot, x) > t] \xrightarrow{t \to \infty} \sigma^i \quad \textit{for all } 1 \leq i \leq d\,.$$

In this case, we can define a Lyapunov spectrum of average expansion rates of the surviving assemble by denoting $\lambda_1 > \lambda_2 > \dots \lambda_p$ for the $0 < p \leq d$ different values of $\sigma^1 \geq \cdots \geq \sigma^d$.

Considering Proposition 5.2.3, the goal would be to show that there are cases where $p > 1$. We will give numerical evidence for this later.

We consider the classical setting without killing at the boundary and replace $(\omega, x) \in \Omega \times \mathbb{R}^d$ by $\omega \in \Omega$ as driving metric dynamics for ease of notation. The Furstenberg-Kesten Theorem [2, Theorem 3.3.3] uses the subadditivity of exterior powers to show convergence of the exponents expressed as singular values. We denote the k-fold exterior powers of the matrix Φ by $\Lambda^k \Phi$ for $1 \leq k \leq d$. Then the condition

$$\sup_{0 \leq t \leq 1} \ln^+ \|\Phi(t, \omega)^{\pm 1}\| \in L^1(\Omega)$$

guarantees that Kingman's Subadditive Ergodic Theorem [2, Theorem 3.3.2] can be used to show that there is a measurable map γ^k such that almost surely

$$\lim_{t \to \infty} \frac{1}{t} \ln \|\Lambda^k \Phi(t, \omega)\| = \gamma^k(\omega)\,.$$

In particular, we obtain

$$\lim_{t \to \infty} \frac{1}{t} \mathbb{E}\left[\ln \|\Lambda^k \Phi(t, \cdot)\|\right] = \mathbb{E}[\gamma^k]\,.$$

We can then define measurable $\sigma^1(\omega) \geq \sigma^2(\omega) \geq \cdots \geq \sigma^d(\omega)$ such that for all $1 \leq k \leq d$ and

$\omega \in \Omega$

$$\sigma^1(\omega) + \sigma^2(\omega) + \cdots + \sigma^k(\omega) = \gamma^k(\omega)\,.$$

Recall that $\sigma_k(\omega)$ denotes the k-th largest singular value of $\Phi(t,\omega)$. By using the well-known fact for all $t > 0$ and $\omega \in \Omega$

$$\sigma_1(\omega) + \cdots + \sigma_k(\omega) = \|\Lambda^k \Phi(t,\cdot)\|\,,$$

one can deduce recursively that

$$\sigma^k(\omega) = \lim_{t\to\infty} \frac{1}{t} \ln \sigma_k(Y(t,\omega))\,.$$

In particular, we may conclude that

$$\mathbb{E}[\sigma^k(\cdot)] = \lim_{t\to\infty} \frac{1}{t} \mathbb{E}\left[\ln \sigma_k(Y(t,\cdot))\right].$$

The different numbers $\lambda_1(\omega) > \cdots > \lambda_{p(\omega)}$ in the sequence $\sigma^1(\omega) \geq \sigma^2(\omega) \geq \cdots \geq \sigma^d(\omega)$ are the Lyapunov exponents. In an ergodic situation, the Lyapunov exponents and their number $p(\omega)$ are constant over almost all ω and we obtain that $\mathbb{E}[\lambda_k(\cdot)] = \lambda_k \in \mathbb{R}$.

The main ingredient for showing Conjecture 5.2.9 is proving the convergence of the exterior powers. As mentioned above, this is achieved in the classical setting by using the subadditivity of

$$\rho^k(\omega, t) := \ln \|\Lambda^k \Phi(t,\omega)\|$$

and then applying Kingman's Subadditive Ergodic Theorem. The subadditivity follows directly from the cocycle property

$$\Lambda^k \Phi(t+s,\omega) = \Lambda^k \Phi(t, \theta_s \omega) \Lambda^k \Phi(s,\omega).$$

Going back to our problem, we would like to show the existence of

$$\lim_{t\to\infty} \frac{1}{t} \mathbb{E}[\rho^k(\cdot,t) | T > t]$$

under mild conditions. An approach using subadditivity would need to establish that

$$\mathbb{E}[\rho^k(\cdot,t+s) | T > t+s] \leq \mathbb{E}[\rho^k(\cdot,t) | T > t] + \mathbb{E}[\rho^k(\cdot,s) | T > s]\,.$$

Unfortunately, showing this seems very difficult, if not impossible, as it is generally not clear what effect the conditioning on non-absorption has on values of $\rho^k(\cot, t)$. Another even more general approach would be to investigate the limit of

$$\mathbb{E}[g(\cdot,t) | T > t]\,,$$

for $g : C([0,\infty)) \times \mathbb{R}_0^+ \to \mathbb{R}$ which satisfy

$$(\omega, t) \mapsto h_t(X_{0\le s\le t}(\omega)) \text{ for maps } h_t : C([0,t]) \to \mathbb{R}\,.$$

But this seems an even harder problem.

The following Lemma shows that we can bound the Lyapunov spectrum, if it exists, from above and below. In more detail, we define an upper and a lower conditioned average Lyapunov exponent λ^u and λ^l by

$$\lambda^u := \limsup_{t\to\infty} \sup_{\|v\|=1} \frac{1}{t}\mathbb{E}\left[\ln \|\mathrm{D}\varphi(t,\cdot,x)v\| | \tilde{T}(\cdot,x) > t\right]\,,$$

and

$$\lambda^l := \liminf_{t\to\infty} \inf_{\|v\|=1} \frac{1}{t}\mathbb{E}\left[\ln \|\mathrm{D}\varphi(t,\cdot,x)v\| | \tilde{T}(\cdot,x) > t\right]\,.$$

Defining similarly to chapter **??**

$$\lambda^+(x) = \max_{\|r\|=1} (\mathrm{D}f(x)r, r),\ \ \lambda^-(x) = \min_{\|r\|=1} (\mathrm{D}f(x)r, r)\,,$$

we find the following bounds for these quantities.

Lemma 5.2.10. *Let (θ,φ) be the random dynamical system with killing at the boundary associated to* (5.2.3)*. Then the upper and lower conditioned average Lyapunov exponents satisfy*

$$\int_X \lambda^-(x)m(\mathrm{d}x) \le \lambda^l \le \lambda \le \lambda^u \le \int_X \lambda^+(x)m(\mathrm{d}x)\,,$$

where λ is the conditioned average Lyapunov exponent given by Proposition 5.2.3.

Proof. For any $\omega \in \Omega$, $x \in E$ and $0 \neq v \in \mathbb{R}^d$ define $r_t(\omega,x,v) := \frac{\mathrm{D}\varphi(t,\omega,x)v}{\|\mathrm{D}\varphi(t,\omega,x)v\|}$. We observe that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|D\varphi(t,\omega,x)v\|^2 &= 2\left(Df(\varphi(t,\omega,x))(t,\omega,x)v, \mathrm{D}\varphi(t,\omega,x)v\right)\\ &= 2\left(\mathrm{D}f(\varphi(t,\omega,x))r_t(\omega,x,v), r_t(\omega,x,v)\right)\|\mathrm{D}\varphi(t,\omega,x)v\|^2\\ &\le 2\lambda^+(\varphi(t,\omega,x))\|\mathrm{D}\varphi(t,\omega,x)v\|^2.\end{aligned}$$

Analogously we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|D\varphi(t,\omega,x)v\|^2 \ge 2\lambda^-(\varphi(t,\omega,x))\|\mathrm{D}\varphi(t,\omega,x)v\|^2\,.$$

Hence, we can conclude that for all $0 \neq v \in \mathbb{R}^d$

$$\|\mathrm{D}\varphi(t,\omega,x)v\|^2 \le \|v\|^2 \exp\left(2\int_0^t \lambda^+(\varphi(s,\omega,x))\mathrm{d}s\right) \tag{5.2.11}$$

$$\|\mathrm{D}\varphi(t,\omega,x)v\|^2 \ge \|v\|^2 \exp\left(2\int_0^t \lambda^-(\varphi(s,\omega,x))\mathrm{d}s\right)\,. \tag{5.2.12}$$

Since λ^+ and λ^- are measurable and bounded on E, we can conclude with Theorem 5.1.5 that

$$\begin{aligned}\lambda^u &\le \limsup_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t \lambda^+(\varphi(s,\cdot,x))\mathrm{d}s|\tilde{T}(\cdot,x) > t\right] \\ &= \lim_{t\to\infty} \mathbb{E}_x\left[\frac{1}{t}\int_0^t \lambda^+(X_s)\,\mathrm{d}s|T > t\right] = \int_X \lambda^+(x)\,m(\mathrm{d}x)\,,\end{aligned}$$

and

$$\begin{aligned}\lambda^l &\ge \liminf_{t\to\infty} \mathbb{E}\left[\frac{1}{t}\int_0^t \lambda^-(\varphi(s,\cdot,x))\mathrm{d}s|\tilde{T}(\cdot,x) > t\right] \\ &= \lim_{t\to\infty} \mathbb{E}_x\left[\frac{1}{t}\int_0^t \lambda^-(X_s)\,\mathrm{d}s|T > t\right] = \int_X \lambda^-(x)\,m(\mathrm{d}x)\,.\end{aligned}$$

The fact that $\lambda^l \le \lambda \le \lambda^u$ follows directly from the respective definitions. □

Note that the computation of λ can be very difficult and costly in higher dimensions as we have to determine $\tilde{m}(\mathrm{d}s,\mathrm{d}x)$, the joint quasi-ergodic distribution of the original process and the derivative angular process. In some cases it will definitely be easier and cheaper to simply compute or approximate $m(\mathrm{d}x)$, the quasi-ergodic distribution of the singular process. Then the integrals $m(\lambda^-)$ and $m(\lambda^+)$ can help to reveal if λ, which can be expected to equal λ_1 in case the spectrum exists, is positive or negative respectively.

5.3 Local synchronisation for nonlinear systems

This section is dedicated to showing implications of λ having a negative sign. As one would expect, the main implication in terms of the random dynamics is the synchronisation of surviving trajectories starting close enough to each other. In the following, we prove a general statement about local synchronisation in discrete time for systems with absorption at the boundary and then show a corollary for systems induced by stochastic differential equations using the definitions and results of the previous two sections.

Theorem 5.3.1 (Local synchronisation theorem)**.** *Let (θ,φ) be a continuously differentiable RDS with killing on a bounded domain $E \subset \mathbb{R}^d$, and let there exist a $\lambda < 0$ such that uniformly over all $x \in E$ and $v \in \mathbb{S}^{d-1}$*

$$\lim_{n\to\infty} \mathbb{P}\left[\frac{1}{n}\ln\|\mathrm{D}\varphi(n,\cdot,x)v\| \le \lambda \middle| \tilde{T}(\cdot,x) > n\right] = 1\,. \tag{5.3.1}$$

Then for all $\lambda_\varepsilon \in (\lambda,0)$, $x \in E$ and $0<\rho<1$, there are $\alpha_x > 0$, $0<\beta<1$, $K_\varepsilon > 1$ and sets $\Omega_x^n \subset \Omega$ with $\mathbb{P}_x(\Omega_x^n|T > n) > 1-\rho$ for all $n \in \mathbb{N}$ such that we have the following:

a) For all $n \in \mathbb{N}$, $\omega \in \Omega_x^n$ and $y, y' \in \overline{B}_{\alpha_x}(x)$

$$\|\varphi(n,\omega,y) - \varphi(n,\omega,y')\| \le K_\varepsilon e^{\lambda_\varepsilon n}\|y-y'\|\,,$$

and, in particular, for all $y \in \overline{B}_{\alpha_x}(x)$

$$\|\varphi(n,\omega,x) - \varphi(n,\omega,y)\| \le \beta e^{\lambda_\varepsilon n}\,.$$

b) *There is exponentially fast local synchronisation of trajectories in discrete time with arbitrarily high probability, i.e.*

$$\lim_{n\to\infty} \mathbb{P}\left(\frac{1}{n}\ln\|\varphi(n,\cdot,x)-\varphi(n,\cdot,y)\| \le \lambda_\varepsilon \ \text{ for all } y \in \overline{B}_{\alpha_x}(x)\middle|\tilde{T}(\cdot,x) > n\right) > 1-\rho\,.$$

Proof. Let $0 < \rho < 1$ be fixed for the following. For $x \in E$ and $n \in \mathbb{N}$, let $\Omega_x^n \subset \{\omega \in \Omega : \tilde{T}(\omega,x) > n\}$ be a set with $\mathbb{P}_x(\Omega_x^n | T > n) > 1-\rho$ whose construction is given later in (5.3.4). Furthermore we define for any $x \in E$

$$U_x := \{y \in \mathbb{R}^d \, : \, x + y \in \overline{E}\}\,.$$

For fixed $(\omega,x) \in \Omega \times E$ we define on U_x

$$Z_n((\omega,x),y) := \varphi(n,\omega,y+x) - \varphi(n,\omega,x)\,.$$

Note that in particular $Z_n((\omega,x),0) = 0$ for all n. Define further

$$F_{(\omega,x)}(y) := Z_1((\omega,x),y)$$

and write

$$F^n_{(\omega,x)} = F_{\Theta^{n-1}(\omega,x)} \circ \cdots \circ F_{(\omega,x)}\,.$$

In addition we define

$$L(\omega,x) = \mathrm{D}F_{(\omega,x)}(0) = \mathrm{D}\varphi(1,\omega,x)$$

and for all $n \ge 1$

$$L_n(\omega,x) = L(\Theta_{n-1}(\omega,x))\,.$$

Similarly to [81], let $0 < \eta = -\lambda_\varepsilon/2$. Since the system is C^1 on a bounded domain, we have

$$G := \sup_{(\omega,x)} \|F_{(\omega,x)}\|_{C^1} < \infty\,.$$

Let $\delta > 0$ be given. Choose $0 < \beta < 1$ such that $G\beta e^\eta < \delta$. Take further $\kappa > 1$ such that $\kappa\beta \le 1$ and $G\kappa\beta e^\eta \le \delta$.

Recall that

$$\Omega_x^K \subset \{\omega \in \Omega \, : \, \tilde{T}(\omega,x) > K\}$$

and define

$$S^K(\beta) = \{y \in U_x : \|F^n_{(\omega,x)}(y)\| \le \beta e^{\lambda_\varepsilon n} \ \text{ for all } 0 \le n \le K \text{ and } \omega \in \Omega_x^K\}\,.$$

For $y \in S^K(\kappa\beta)$ we define for all $1 \le n \le K$

$$L'_n(\omega, x) = \int_0^1 \mathrm{D}F_{\Theta_{n-1}(\omega,x)}(tF^{n-1}_{(\omega,x)}(y))\,\mathrm{d}t\,.$$

Observe that this choice yields for $1 \le n \le K$

$$L'^n(\omega, x)y = L'_n(\omega, x)\cdots L'_1(\omega, x)y = F^n_{(\omega,x)}(y)\,.$$

We deduce that for any $y \in S^K(\kappa\beta)$

$$\begin{aligned}\sup_{\omega\in\Omega^K_x}\sup_{n\le K}\|L'_n(\omega, x) - L_n(\omega, x)\|e^{\eta n} &\le \sup_{\omega\in\Omega^K_x}\sup_{n\le K}\|\mathrm{D}F_{\theta_{n-1}(\omega,x)}\|\kappa\beta\exp\Big(n(\eta+\lambda_\varepsilon) - \lambda_\varepsilon\Big)\\ &\le G\kappa\beta e^{\eta} < \delta\,.\end{aligned}$$

Claim: We can deduce from

$$\sup_{\omega\in\Omega^K_x}\sup_{n\le K}\|L'_n(\omega, x) - L_n(\omega, x)\|e^{\eta n} < \delta$$

that for all $\omega \in \Omega^K_x$, $y \in S^K(\kappa\beta)$ and $1 \le n \le K$

$$\|L'^n(\omega, x)y\| \le K_\varepsilon e^{n\lambda_\varepsilon}\|y\|$$

uniformly over K, for some $K_\varepsilon > 1$.

Let us assume that the claim is true for the time being and define $d_x = d(x, \partial E)/2$. Choose $\alpha_x = \min(d_x, \beta/K_\varepsilon) < \beta$. From the claim we observe that uniformly over K for all $y \in \overline{B}_{\alpha_x}(0) \cap S^K(\kappa\beta)$ and $1 \le n \le K$

$$\|F^n_{(\omega,x)}(y)\| \le K_\varepsilon e^{n\lambda_\varepsilon}\alpha_x \le \beta e^{n\lambda_\varepsilon}$$

and therefore

$$D_K(\alpha_x) := \overline{B}_{\alpha_x}(0) \cap S^K(\beta) = \overline{B}_{\alpha_x}(0) \cap S^K(\kappa\beta)\,.$$

Since the boundaries of $S^K(\beta)$ and $S^K(\kappa\beta)$ are disjoint, this implies that $\overline{B}_{\alpha_x}(0) = D_K(\alpha_x)$ for all $K > 0$ and the second statement in a) follows.

For any $y, y' \in D_K(\alpha_x)$ and $1 \le n \le K$ we define similarly to before

$$L'_n((\omega, x)) = \int_0^1 \mathrm{D}F_{\Theta_{n-1}(\omega,x)}\left(tF^{n-1}_{(\omega,x)}(y) + (1-t)F^{n-1}_{(\omega,x)}(y')\right)\mathrm{d}t\,.$$

Observe that for $1 \le n \le K$

$$L'^n(\omega, x)(y - y') = L'_n(\omega, x)\cdots L'_1(\omega, x)(y - y') = F^n_{(\omega,x)}(y) - F^n_{(\omega,x)}(y')\,.$$

Therefore we observe analogously to before that

$$\sup_{\omega \in \Omega_x^K} \sup_{n \leq K} \|L'_n(\omega, x) - L_n(\omega, x)\| e^{\eta n} < \delta \,,$$

which using the claim gives that for all $\omega \in \Omega_x^K$, $y, y' \in D_K(\alpha_x)$ and $1 \leq n \leq K$

$$\|F^n_{(\omega,x)}(y) - F^n_{(\omega,x)}(y')\| \leq K_\varepsilon e^{\lambda_\varepsilon n} \|y - y'\| \,.$$

Hence, the first statement in a) follows.

Now, we prove the claim above. To make the notation easier, we fix (ω, x) and write L for the Jacobian and L' for the perturbation. Take $\|\xi^{(0)}\| = 1$, define the sequence $\xi^{(n)}$ as units along L_n and write

$$L_n \xi^{(n-1)} = t^{(n)} \xi^{(n)} \,.$$

For our purposes it is enough to exercise through the one-dimensional case since the convergence in (5.3.1) is assumed to be uniform over all $v \in \mathbb{S}^{d-1}$ in the d-dimensional scenario. We refer to [81] for the full details of the multidimensional scenario which are not needed here.

If we write $L'^n u = u^{(n)}$, we see that $\sup_n \|L'_n - L_n\| e^{\eta n} < \delta$ implies

$$\left| u^{(n)} \right| \leq t^{(n)} \left| u^{(n-1)} \right| + \delta e^{-n\eta} \left| u^{(n-1)} \right| \,. \tag{5.3.2}$$

Since the finite-time Lyapunov exponents are bounded away from $-\infty$ according to the assumptions of the model, there is a $C > 0$ independent from (ω, x) such that for any $N \in \mathbb{N}$

$$\frac{1}{C} e^{-N\eta} \leq t^{(N)} \,.$$

If we fix $\nu \geq 1$ and set $U^{(\nu)} = \left| u^{(\nu)} \right|$ and for $N > \nu$

$$U^{(N)} = \left(\prod_{n=\nu+1}^{N} t^{(n)} \right) \left(\prod_{n=\nu+1}^{N} (1 + C\delta e^{-n\eta}) \right) U^{(\nu)} \,,$$

we can observe with (5.3.2) that $U^{(N)} \geq \left| u^{(N)} \right|$ for all $N \geq \nu$.

Now we set $\delta = \frac{1}{C} \prod_{n=1}^{\infty} (1 - e^{-n\eta})^2$ and

$$C' = \frac{\prod_{n=1}^{\infty} (1 + C\delta e^{-n\eta})}{\prod_{n=1}^{\infty} (1 - e^{-n\eta})} \leq \prod_{n=1}^{\infty} (1 - e^{-n\eta})^{-2} = \frac{1}{C\delta} \,.$$

Note that δ and C' do not depend on (ω, x). It is easy to infer similarly to [81] that

$$\left| u^{(N)} \right| \leq C' \left(\prod_{n=\nu+1}^{N} t^{(n)} \right) \left(\prod_{n=\nu+1}^{N} (1 - e^{-n\eta}) \right) \left| u^{(\nu)} \right| \tag{5.3.3}$$

and

$$\left|u^{(N)}\right| \geq \left(\prod_{n=\nu+1}^{N} t^{(n)}\right)\left(\prod_{n=\nu+1}^{N}(1-e^{-n\eta})\right)\left|u^{(\nu)}\right| .$$

We take $\nu = 0$ from here on. Observe that the finite Lyapunov exponents satisfy

$$\lambda(N,\omega,x) = \frac{1}{N}\ln\prod_{n=1}^{N} t^{(n)} .$$

Let $\varepsilon = \lambda_\varepsilon - \lambda > 0$. By assumption, there exists an $N^* > 0$ such that for all $N \geq N^*$ we have

$$\mathbb{P}\left(\lambda(N,\cdot,x) < \lambda + \varepsilon/2 \middle| \tilde{T}(\cdot,x) > N\right) > 1-\rho .$$

Recall that δ, C, C' do not depend on (ω, x). Now define the measurable sets

$$\Omega_x^N = \begin{cases} \{\omega \in \Omega \,:\, \tilde{T}(\omega,x) > N,\ \lambda(N,\omega,x) < \lambda+\varepsilon/2\} & \text{if } N \geq N^* , \\ \{\omega \in \Omega \,:\, \tilde{T}(\omega,x) > N\} & \text{if } N < N^* . \end{cases} \tag{5.3.4}$$

Hence, $\mathbb{P}_x(\Omega_x^n | T > n) > 1-\rho$ for all $n \in \mathbb{N}$. We can conclude from (5.3.3) that there is $C'' > 0$ such that for all $x \in E$, $N \geq N^*$ and $\omega \in \Omega_x^N$

$$\frac{1}{N}\ln\|L'^N(\omega,x)\| \leq \frac{C''}{N} + \frac{1}{N}\ln\prod_{n=\nu+1}^{N} t^{(n)} = \frac{C''}{N} + \lambda(N,\omega,x) .$$

From the assumptions we have that

$$s^* := \sup_{(\omega,x)}\sup_{N\leq N^*}\lambda(N,\omega,x) < \infty .$$

We define

$$K_\varepsilon = \max\{e^{C''}, e^{-\lambda_\varepsilon N^*} s^*\} .$$

Then we obtain the statement of the claim, i.e. for all $N \in \mathbb{N}$ and $\omega \in \Omega_x^N$

$$\|L'^N(\omega,x)\| \leq K_\varepsilon e^{\lambda_\varepsilon N} .$$

Finally, we show statement b). We conclude from the second statement in a), and the fact that $\beta < 0$, that for all $n \in \mathbb{N}$

$$\begin{aligned} &\mathbb{P}\left(\|\varphi(n,\cdot,x) - \varphi(n,\cdot,y)\| \leq e^{\lambda_\varepsilon n} \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot,x) > n\right) \\ &\geq \mathbb{P}\left(\Omega_x^n \middle| \tilde{T}(\cdot,x) > n\right) > 1-\rho . \end{aligned}$$

Hence, we actually obtain for all $n \in \mathbb{N}$

$$\mathbb{P}\left(\frac{1}{n}\ln\|\varphi(n,\omega,x)-\varphi(,\omega,y)\| \leq \lambda_\varepsilon \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot,x) > n\right) > 1-\rho\,,$$

which implies the claim. □

We can immediately observe the following corollary about local synchronisation of random dynamical systems induced by solutions of stochastic differential equations with absorption at the boundary of a domain:

Corollary 5.3.2. *Let the RDS with killing induced by the SDE* (5.2.3) *have a negative conditioned asymptotic Lyapunov exponent* $\lambda < 0$. *Then for all* $x \in E$, $\lambda_\varepsilon \in (\lambda, 0)$ *and* $0 < \rho < 1$, *there is an* $\alpha_x > 0$ *such that*

$$\lim_{n\to\infty}\mathbb{P}\left(\frac{1}{n}\ln|\varphi(n,\cdot,x)-\varphi(n,\cdot,y)| \leq \lambda_\varepsilon \text{ for all } y \in \overline{B}_{\alpha_x}(x) \middle| \tilde{T}(\cdot,x) > n\right) > 1-\rho.$$

Proof. This is an immediate consequence of Theorem 5.3.1 in combination with Theorem 5.2.8. □

We would like to show a global synchronisation theorem as well. An obvious very weak statement is expressed in the following proposition which is simply supposed to sketch a first direction for this endeavour:

Proposition 5.3.3. *Let the RDS with killing induced by the SDE* (5.2.3) *have a negative conditioned average Lyapunov exponent* λ. *Let* $x, y \in I$, α_x *be the radius of stability from Theorem 5.3.1 for some* $0 < \rho < 1$ *and let there be a time* $t \geq 0$ *such that*

$$\mathbb{P}\left(|\varphi(t,\cdot,x)-\varphi(t,\cdot,y)| < \alpha_x \middle| \tilde{T}(\cdot,x) > t\right) > 0.$$

Then there is a sequence $t_n \to \infty$ *such that for any* $\varepsilon > 0$

$$\lim_{n\to\infty}\mathbb{P}\left(|\varphi(t_n,\cdot,x)-\varphi(t_n,\cdot,y)| < \varepsilon \middle| \tilde{T}(\cdot,x) > t_n\right) > 0.$$

Proof. This is a direct consequence of Corollary 5.3.2 in combination with the Markov property. □

In the one-dimensional case one could hope for using monotonicity arguments to show a stronger statement. This is left as future work.

5.4 Relation to pathwise random dynamics

This section is dedicated to finding further relations between Markov processes with absorption at the boundary and the induced random dynamical systems, in particular understanding the latter ones as skew product systems. It suggests itself to investigate the connection between quasi-stationary and quasi-ergodic measures and suitable associated measures for the skew product, similarly to the setting without killing as introduced in Chapter 1. Hereby, conditionally invariant measures for dynamical systems with a hole turn out to be the he right concept.

5.4.1 Conditionally invariant measures on the skew product space

Let $(X_t)_{t\geq 0}$ be a Markov process on some topological space X, defined over a filtered probability space $(\Omega^+, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and equipped with a family of maps $\theta_t : \Omega^+ \to \Omega^+$ such that

$$\theta_{t+s} = \theta_t \circ \theta_s \quad \text{for all } t, s \geq 0$$

and

$$X_{t+h}(\omega) = X_t(\theta_h \omega) \quad \text{for all } t, h \geq 0 \text{ and } \omega \in \Omega^+ .$$

Note that the existence of (θ_t) is sometimes part of the definition of a Markov process (see e.g. [85]) and that the σ-algebra generated by θ_t is independent from $\mathcal{F}_t$ for all $t \geq 0$. Let further θ_t be $\mathbb{P}$-invariant for all $t \geq 0$ and the process induce a random dynamical system with one-sided skew product flow

$$\Theta_t : \Omega^+ \times X \to \Omega^+ \times X, \ \Theta_t(\omega, x) = (\theta_t \omega, \varphi(t, \omega, x)) , \quad \text{for all } t \geq 0 .$$

As before let $E \subset X$ be some open subset such that $(X_t)_{t\geq 0}$ with transition probabilities $(\mathbb{P}_x)_{x\in E}$ satisfies the assumptions of this chapter if the process is absorbed at ∂E. Note that the described setting is for example given in the situation of random dynamical systems induced by stochastic differential equations as described in Section 5.1.2, defined on the one-sided canonical path space with Wiener measure $\mathbb{P}$ projected to one-sided time and canonical one-sided shift $(\theta_t)_{t\geq 0}$.

We now consider the problem of killed diffusion from the perspective of dynamical systems with holes. We can write $E = \mathring{X} = X \setminus H$ where $H = X \setminus E$ is a closed set constituting a hole in the space X. Define $\mathring{M} = \Omega^+ \times \mathring{X} = \Omega^+ \times E$ as the product space with hole in only one factor and

$$\mathring{M}_t = \bigcap_{0\leq s\leq t} (\Theta_s)^{-1} \mathring{M} , \quad \mathring{\Theta}_t = \Theta_t\big|_{\mathring{M}_t} , \tag{5.4.1}$$

i.e. the survival set and the flow restricted to the survival set. We further define for each $\omega \in \Omega^+$

$$E_t(\omega) = \bigcap_{0\leq s\leq t} \varphi(s, \omega, \cdot)^{-1} E , \quad \mathring{\varphi}(t, \omega, \cdot) = \varphi(t, \omega, \cdot)\big|_{E_t} ,$$

i.e. the ω-wise survival set and cocycle restricted to the survival set.

Definition 5.4.1 (Conditionally invariant measure)**.** A measure μ supported on $\mathring{M}$ is called a *conditionally invariant probability measure* for $(\mathring{\Theta}_t)_{t\geq 0}$ on $\Omega^+ \times E$ if

$$\frac{\mu\left(\mathring{\Theta}_t^{-1}(C)\right)}{\mu(\mathring{M}_t)} = \mu(C) \quad \text{for all } t \geq 0 \text{ and } C \in \mathcal{F} \times \mathcal{B}(E)\,.$$

See for example [29] or [34] for fundamental work on conditionally invariant probability measures. First, we can show the following result similarly to Homburg and Zmarrou [50, Lemma 5.2.].

Proposition 5.4.2. *The measure $\mathbb{P} \times \nu$ is a conditionally invariant probability measure for the one-sided skew product flow $(\mathring{\Theta}_t)_{t\geq 0}$ on $\Omega^+ \times E$ iff the measure v is a quasi-stationary distribution for $(X_t)_{t\geq 0}$ on E.*

Proof. The measure v being quasi-stationary means that for all Borel-sets $B \subset E$ we have

$$\nu(B) = \mathbb{P}_\nu\left(X_t \in B \middle| T > t\right) = \frac{\int_E \mathbb{P}(\varphi(t,\cdot,x) \in B, \tilde{T}(\cdot,x) > t)\,\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x(T > t)\,\nu(\mathrm{d}x)}\,. \tag{5.4.2}$$

Let $C \subset \Omega^+ \times E$ be measurable with respect to $\mathcal{F} \times \mathcal{B}(E)$. Hence, we may assume $C = A \times B$. Let us first assume that ν is a quasi-stationary distribution for the process. We observe that

$$\frac{(\mathbb{P}\times\nu)\left(\mathring{\Theta}_t^{-1}C\right)}{(\mathbb{P}\times\nu)\mathring{M}_t} = \frac{\int_E \mathbb{P}\left(\omega\,:\,\theta_t\omega \in A, \varphi(t,\omega,x) \in B, \tilde{T}(\omega,x) > t\right)\,\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T > t\right)\,\nu(\mathrm{d}x)}\,.$$

Recall that $(\varphi(t,\cdot,\cdot))_{t\geq 0}$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and $\sigma(\theta_t)$ is independent from $\mathcal{F}_t$ for all $t \geq 0$ since θ_t is the one-sided shift on Ω^+. Hence, we can infer with (5.4.2) that

$$\begin{aligned}\frac{(\mathbb{P}\times\nu)\left(\mathring{\Theta}_t^{-1}C\right)}{(\mathbb{P}\times\nu)\mathring{M}_t} &= \frac{\int_E \mathbb{P}(\omega\,:\,\theta_t\omega \in A)\mathbb{P}\left(\omega\,:, \varphi(t,\omega,x) \in B, \tilde{T}(\omega,x) > t\right)\,\nu(\mathrm{d}x)}{\int_E \mathbb{P}_x\left(T > t\right)\,\nu(\mathrm{d}x)} \\ &= \mathbb{P}_\nu\left(X_t \in B \middle| T > t\right)\mathbb{P}(A) = \nu(B)\mathbb{P}(A) = (\mathbb{P}\times\nu)(C)\,.\end{aligned} \tag{5.4.3}$$

As one can see directly from (5.4.3), we can show the reverse direction analogously, i.e. assuming $\mathbb{P} \times \nu$ to be conditionally invariant immediately gives us (5.4.2) and therefore the quasi-stationarity of ν. □

Recall from the proof of Theorem 5.1.5 that for any bounded and measurable function f, $x \in E$ and $0 < p < 1$

$$\lim_{t\to\infty} \mathbb{E}_x(f(X_{pt})|T > t) = \int_I f(x)\,m(\mathrm{d}x)\,.$$

This observation prompts the definition of the maps $H_t^p : \Omega^+ \times X \to \Omega^+ \times X$ by

$$H_t^p(\omega,x) = (\theta_t\omega, \varphi(pt,\omega,x)) \;\text{ for all } \omega \in \Omega^+, x \in X\,,$$

for all $t \geq 0$ and $0 < p < 1$. This means that the shift on the probability space is considered for time $t \geq 0$ whereas the mapping on the state space is only followed up to time $0 \leq pt < t$. Note that H_t^p does not satisfy the flow property but the definition is supposed to illustrate the distinction between quasi-stationarity and quasi-ergodicity. We give the following definition:

Definition 5.4.3. Let $0 < p < 1$. A *p-quasi-stationary distribution* (QSD) for the process $(X_t)_{t\geq 0}$ is a probability measure m on E such that for all $t \geq 0$ and measurable sets $B \subset E$

$$\mathbb{P}_m \left(X_{pt} \in B | T > t \right) = m(B) \,. \tag{5.4.4}$$

As before, set $\mathring{H}_t^p = H_t^p|_{\mathring{M}_t}$. We can prove the following proposition analogously to Proposition 5.4.2:

Proposition 5.4.4. *For any $0 < p < 1$, the measure $\mathbb{P}\times m$ is a conditionally invariant probability measure for the family of one-sided skew product maps $(\mathring{H}_t^p)_{t\geq 0}$ iff the measure m is a p-quasi-stationary distribution for $(X_t)_{t\geq 0}$ on E.*

Proof. Recall from Definition 5.4.4 that the measure m is a p-quasi-stationary probability distribution iff for all Borel-sets $B \subset E$ we have

$$m(B) = \mathbb{P}_m \left(X_{pt} \in B \middle| T > t \right) = \frac{\int_E \mathbb{P}(\varphi(pt, \cdot, x) \in B, \tilde{T}(\cdot, x) > t) \, m(\mathrm{d}x)}{\int_E \mathbb{P}_x(T > t) \, m(\mathrm{d}x)} \,. \tag{5.4.5}$$

Let $C \subset \Omega^+ \times E$ be measurable with respect to $\mathcal{F} \times \mathcal{B}(E)$. Hence, we may assume $C = A \times B$. Let us first assume that m is a p-quasi-stationary distribution for the process. We observe that

$$\frac{(\mathbb{P} \times m) \left((\mathring{H}_t^p)^{-1} C \right)}{(\mathbb{P} \times m) \mathring{M}_t} = \frac{\int_E \mathbb{P} \left(\omega \, : \, \theta_t \omega \in A, \varphi(pt, \omega, x) \in B, \tilde{T}(\omega, x) > t \right) \, m(\mathrm{d}x)}{\int_E \mathbb{P}_x \left(T > t \right) \, m(\mathrm{d}x)} \,.$$

Recall that $(\varphi(pt, \cdot, \cdot))_{t\geq 0}$ is adapted to $(\mathcal{F}_{pt})_{t\geq 0}$ and $\sigma(\theta_t)$ is independent from $\mathcal{F}_{pt}$ for all $t \geq 0$ since θ_t is the one-sided shift on Ω^+ and $p < 1$. Hence, we can infer with (5.4.5) that

$$\begin{aligned} \frac{(\mathbb{P} \times m) \left(\mathring{\Theta}_t^{-1} C \right)}{(\mathbb{P} \times m) \mathring{M}_t} &= \frac{\int_E \mathbb{P}(\omega \, : \, \theta_t \omega \in A) \mathbb{P} \left(\omega \, :, \varphi(pt, \omega, x) \in B, \tilde{T}(\omega, x) > t \right) \, m(\mathrm{d}x)}{\int_E \mathbb{P}_x \left(T > t \right) \, m(\mathrm{d}x)} \\ &= \mathbb{P}_m \left(X_{pt} \in B \middle| T > t \right) \mathbb{P}(A) = m(B) \mathbb{P}(A) = (\mathbb{P} \times m)(C) \,. \end{aligned} \tag{5.4.6}$$

As one can see directly from (5.4.6), we can show the reverse direction analogously, i.e. assuming $\mathbb{P} \times m$ to be conditionally invariant for $(\mathring{H}_t^p)_{t\geq 0}$ immediately gives us (5.4.5) and therefore that m is a p-quasi-stationary distribution for the process $(X_t)_{t\geq 0}$. □

Making Proposition 5.4.4 useful for characterising a quasi-ergodic distribution m requires p-quasi-stationarity of m for some $0 < p < 1$. We conjecture the following even stronger statement for which we have not found a proof yet. Such a proof should work similarly to [74, Proposition 1], using the same tricks as in the proof of Theorem 5.1.5.

Conjecture 5.4.5. *If m is a quasi-ergodic distribution for $(X_t)_{t\geq 0}$ on E, then it is also a p-quasi-stationary distribution for any $0 < p < 1$.*

Therefore we obtain as a direct consequence of Proposition 5.4.4: if the measure m is a quasi-ergodic distribution for $(X_t)_{t\geq 0}$ on E, the measure $\mathbb{P} \times m$ is a conditionally invariant probability measure for the family of one-sided skew product maps $(\mathring{H}_t^p)_{t\geq 0}$ for any $0 < p < 1$.

We have discussed the correspondences between, on the one side, quasi-stationary and quasi-ergodic distributions for the process and, on the other side, conditionally invariant measures for the skew product systems, as long as we consider everything in one-sided time. Usually, as introduced in Chapter 1, the metric dynamical system on the probability space, underlying the random dynamical system, is considered in two-sided time. This is necessary if we want to investigate random attractors, for example.

5.4.2 Two-sided time and relation to the survival process

We now consider the problem in two-sided time, i.e. we consider $(\Omega, \mathcal{F}, (\mathcal{F}_s^t)_{s\leq t\in\mathbb{R}}, \mathbb{P})$ with the maps $(\theta_t)_{t\in\mathbb{R}}$ such that θ_t is $\mathbb{P}$-invariant for all $t \in \mathbb{R}$. A natural question to ask is whether in this situation there is also a conditionally invariant probability measure for the skew product flow corresponding with a quasi-stationary distribution ν. In more detail, we would like to keep $\mathbb{P}$ fixed as the marginal and find a conditionally invariant measure for $(\mathring{\Theta}_t)_{t\geq 0}$ of the form

$$\mu(\mathrm{d}\omega, \mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)\,, \tag{5.4.7}$$

where the μ_ω are measurable with respect to $\mathcal{F}_{-\infty}^0$, i.e. the past of the system, and

$$\int_\Omega \mu_\omega(\cdot)\,\mathbb{P}(\mathrm{d}\omega) = \nu(\cdot)\,,$$

analogously to the case without killing of trajectories where these measures are called Markov measures. From the previous section we know that if the μ_ω are demanded to be measurable with respect to $\mathcal{F}_0^\infty$, i.e. the future of the system, we get $\mu_\omega = \nu$ almost surely according to Proposition 5.4.2.

Let us first assume the existence of such a conditionally invariant Markov measure. Then we observe that the invariance of the sample measures requires an additional assumption.

Lemma 5.4.6. *Assume that there exists a conditionally invariant Markov probability measure μ on $\Omega \times E$ for $(\mathring{\Theta}_t)_{t\geq 0}$. Then for any $t \in \mathbb{R}^+$, its disintegrations μ_ω such that $E_t(\omega)$ is non-empty satisfy the relation*

$$\frac{\mu_\omega\left(\mathring{\varphi}(t,\omega,\cdot)^{-1}A\right)}{\mu_\omega(E_t(\omega))} = \mu_{\theta_t\omega}(A) \quad \textit{for almost all } \omega \in \Omega \quad \textit{and all } A \in \mathcal{B}(E)\,, \tag{5.4.8}$$

if and only if for all $t \geq 0$ we have

$$\mu_\omega(E_t(\omega)) = \mu(\mathring{M}_t) \quad \textit{for almost all } \omega \in \Omega \textit{ with } E_t(\omega) \neq \emptyset\,. \tag{5.4.9}$$

Proof. Fix $t \geq 0$. We proceed similarly to [61, Proposition 1.3.27]. Take bounded and measurable functions $f : E \to \mathbb{R}$ and $g : \Omega \to \mathbb{R}$ and observe that by definition of the conditional invariance

$$\int_{\Omega \times E} f(x) g(\omega) \mu_\omega(\mathrm{d}x) \mathbb{P}(\mathrm{d}\omega) = \frac{\int_{\Omega \times E} f(\varphi(t,\omega,x)) g(\theta_t \omega) \mathbb{1}_{\mathring{M}_t}(\omega, x) \mu_\omega(\mathrm{d}x) \mathbb{P}(\mathrm{d}\omega)}{\int_\Omega \mu_\omega(E_t(\omega)) \mathbb{P}(\mathrm{d}\omega)} .$$

Using the invariance of $\mathbb{P}$ with respect to θ_t for the change of variables $\omega \to \theta_{-t}\omega$, we can therefore write

$$\begin{aligned} \int_\Omega \left[\int_E f(x) \mu_{\theta_t \omega}(\mathrm{d}x) \right] g(\omega) \mathbb{P}(\mathrm{d}\omega) &= \int_\Omega \left[\int_E f(x) \mu_\omega(\mathrm{d}x) \right] g(\theta_{-t}\omega) \mathbb{P}(\mathrm{d}\omega) \\ &= \frac{\int_\Omega \left[\int_E f(\varphi(t,\omega,x)) \mathbb{1}_{\mathring{M}_t}(\omega, x) \mu_\omega(\mathrm{d}x) \right] g(\omega) \mathbb{P}(\mathrm{d}\omega)}{\int_\Omega \mu_\omega(E_t(\omega)) \mathbb{P}(\mathrm{d}\omega)} . \end{aligned}$$

Since this is true for any g, we conclude that for almost all $\omega \in \Omega$

$$\frac{\int_E f(\varphi(t,\omega,x)) \mathbb{1}_{E_t(\omega)}(x) \mu_\omega(\mathrm{d}x)}{\int_\Omega \mu_\omega(E_t(\omega)) \mathbb{P}(\mathrm{d}\omega)} = \int_E f(x) \mu_{\theta_t \omega}(\mathrm{d}x) .$$

Hence, equation (5.4.8) is satisfied if and only if

$$\mu_\omega(E_t(\omega)) = \int_\Omega \mu_{\omega'}(E_t(\omega')) \mathbb{P}(\mathrm{d}\omega') = \mu(\mathring{M}_t) \;\text{ for almost all } \omega \in \Omega \text{ with } E_t(\omega) \neq \emptyset ,$$

which is equivalent to Assumption (5.4.9). □

Two questions arise. First of all, one is inclined to ask how reasonable Assumption (5.4.9) is. Secondly, one might want to investigate if ν as given in (5.4.7) can now be shown to be a quasi-stationary distribution for the associated Markov process. Both are theoretical questions that should be investigated in the future.

Let us now assume that a quasi-stationary distribution ν exists. The question is how to construct the disintegrations μ_ω such that μ is a conditionally invariant Markov probability measure. The natural analogue to the case without absorption is given by

$$\mu_\omega(\cdot) := \lim_{t \to \infty} \frac{\nu\left(\mathring{\varphi}(t, \theta_{-t}\omega, \cdot)^{-1}(\cdot)\right)}{\nu(E_t(\theta_{-t}\omega))} ,$$

if the limit exists. However, due to our assumption of almost sure killing, the numerator and denominator both become zero in finite time for almost all ω such that this limit cannot exist for any measurable subset of E.

One idea to overcome this problem would be to investigate if the Q-process, i.e. the survival process, corresponds to a particular random dynamical system that can be studied in infinite time in the classical way. However, we explain why this does not seem feasible. Recall from Section 5.1.1 that the Q-process or survival process $(Y_t)_{t \geq 0}$ is the Markov process with state space E, filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ and transition probabilities $(\mathbb{Q}_x)_{x \in E}$ such that for any $s \geq 0$

$$\mathbb{Q}_x((Y_u)_{0\leq u\leq s} \in \cdot) = \lim_{t\to\infty} \mathbb{P}_x((X_u)_{0\leq u\leq s} \in \cdot | T > t)\,.$$

Further the Markov semigroup of operators Q_t defined by $Q_t f(x) = \mathbb{E}_x(f(Y_t))$ on an appropriate function space is m-invariant where m is the quasi-ergodic measure for the killed process. If there was a random dynamical system (θ, φ) associated with this Q-process, each ω-trajectory $\varphi(t, \omega, x) = Y_t(\omega)$ starting from $x \in E$ would require a completely new interpretation since the boundary is never hit. If, for example, the original process $(X_t)_{t\geq 0}$ is a solution process to a stochastic differential equation, the process $(Y_t)_{t\geq 0}$ cannot solve a corresponding stochastic differential equation because there is no hitting of the boundary for the survival process. The possible generation of a random dynamical system could come from a random differential equation with bounded noise that is solved by the process $(Y_t)_{t\geq 0}$. However, it is entirely opaque how such an equation could be derived. A completely abstract generation of a random dynamical system from the Q-process seems even less attainable.

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