Freie Universität Berlin, Department of Mathematics

Lecture Notes on Random Dynamical Systems

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Maximilian Engel

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Main references

The main references for this lecture are [2, 21, 27], but there will be a list of several other references that are updated throughout the course.

Chapter 0

Some elements of measure theory, dynamical systems and functional analysis

0.1 Measures and measure spaces

0.1.1 Basic definitions and properties

We collect the most basic definitions in measure theory, followed by some results which will be useful in the lectures.

Definition 0.1.1 (Algebra and σ-algebra). Consider a collection $\mathcal{A}$ of subsets of a set X with $\emptyset \in \mathcal{A}$, and the following properties:

(a) When $A \in \mathcal{A}$ then $A^c := X \setminus A \in \mathcal{A}$.

(b) When $A, B \in \mathcal{A}$ then $A \cup B \in \mathcal{A}$.

(b') Given a finite or infinite sequence $\{A_k\}$ of subsets of X, $A_k \in \mathcal{A}$, then also $\bigcup_k A_k \in \mathcal{A}$.

If $\mathcal{A}$ satisfies (a) and (b), it is called an *algebra* of subsets of X; if it satisfies (a) and (b'), it is called a *σ-algebra* .

It follows from the definition that a σ-algebra is an algebra, and for an algebra $\mathcal{A}$ holds

- $\emptyset, X \in \mathcal{A}$;
- $A, B \in \mathcal{A} \Rightarrow A \cap B \in \mathcal{A}$;
- $A, B \in \mathcal{A} \Rightarrow A \setminus B \in \mathcal{A}$;
- if $\mathcal{A}$ is a σ-algebra , then $\{A_k\} \subset \mathcal{A} \Rightarrow \bigcap_k A_k \in \mathcal{A}$.

Definition 0.1.2 (Measure). A function $\mu : \mathcal{A} \to [0, \infty]$ on a σ-algebra $\mathcal{A}$ is a *measure* if

(a) $\mu(\emptyset) = 0$;

(b) $\mu(A) \geq 0$ for all $A \in \mathcal{A}$; and

(c) $\mu\left(\bigcup_k A_k\right) = \sum_k \mu(A_k)$ if $\{A_k\}$ is a finite or infinite sequence of pairwise disjoint sets from $\mathcal{A}$, that is, $A_i \cap A_j = \emptyset$ for $i \neq j$. This property of μ is called *σ-additivity* (or *countable additivity*).

If, in addition, $\mu(X) = 1$, then μ is called a *probability measure.*

Definition 0.1.3.

(a) If $\mathcal{A}$ is a σ-algebra of subsets of X and μ is a measure on $\mathcal{A}$, then the triple $(X, \mathcal{A}, \mu)$ is called a *measure space.* The subsets of X contained in $\mathcal{A}$ are called *measurable.*

(b) If $\mu(X) < \infty$ (resp. $\mu(X) = 1$) then the measure space is called *finite* (resp. *probabilistic* or *normalized*).

(c) If there is a sequence $\{A_k\} \subset \mathcal{A}$ satisfying $X = \bigcup_k A_k$ and $\mu(A_k) < \infty$ for all k, then the measure space $(X, \mathcal{A}, \mu)$ is called *σ-finite.*

A set $N \in \mathcal{A}$ with $\mu(N) = 0$ is called a *null set.* If a certain property involving the points of a measure space holds true except for a null set, we say the property holds *almost everywhere* (we write a.e., which, depending on the context, sometimes means "almost every"). We also use the word *essential* to indicate that a property holds a.e. (e.g., "essential bijection").

Definition 0.1.4. The σ-algebra generated by a collection $\mathcal{A}_0$ of subsets of X, also denoted by $\sigma(\mathcal{A}_0)$, is the smallest σ-algebra containing $\mathcal{A}_0$, i.e.

$$\sigma(\mathcal{A}_0) = \bigcap_{\mathcal{A} \text{ is a } \sigma\text{-algebra with } \mathcal{A}_0 \subseteq \mathcal{A}} \mathcal{A}.$$

Given two measurable spaces $(X_1, \mathcal{A}_1)$ and $(X_2, \mathcal{A}_2)$, the σ-algebra generated by the products of subsets of X_1 and X_2, i.e.,

$$A_1 \otimes A_2 := \sigma(\{A_1 \times A_2 \, : \, A_1 \in \mathcal{A}_1, A_2 \in \mathcal{A}_2\})$$

is called the product σ-algebra .

Analogously we can define the algebra of subsets of X generated by some collection of subsets of X.

Theorem 0.1.5 (Hahn–Kolmogorov extension theorem)**.** *Let X be a set, $\mathcal{A}_0$ an algebra of subsets of X, and $\mu_0 : \mathcal{A}_0 \to [0, \infty]$ a σ-additive function. If $\mathcal{A}$ is the σ-algebra generated by $\mathcal{A}_0$, there exists a measure $\mu : \mathcal{A} \to [0, \infty]$ such that $\mu\big|_{\mathcal{A}_0} = \mu_0$. If μ_0 is σ-finite, the extension is unique.*

This result becomes especially useful if we would like to define measures on sets of sequences.

Definition 0.1.6 (Cylinder)**.** Let $\mathcal{A}_k$ be a σ-algebra for $k \in \mathrm{N}$. Let $k_1 < k_2 < \ldots < k_r$ be integers and $A_{k_i} \in \mathcal{A}_{k_i}$, $i = 1, \ldots, r$. A *cylinder set* (also called *rectangle*) is a set of the form

$$[A_{k_1}, \ldots, A_{k_r}] = \{\{x_j\}_{j \in \mathrm{N}} : x_{k_i} \in A_{k_i},\ 1 \le i \le r\}.$$

Definition 0.1.7. Let $(X_i, \mathcal{A}_i, \mu_i)$, $i \in \mathrm{N}$, be normalized measure spaces. The *product measure space* $(X, \mathcal{A}, \mu) = \prod_{i \in \mathrm{N}} (X_i, \mathcal{A}_i, \mu_i)$ is defined by

$$X = \prod_{i \in \mathrm{N}} X_i \quad \text{and} \quad \mu\left([A_{k_1}, \ldots, A_{k_r}]\right) = \prod_{j=1}^{r} \mu_{k_j}(A_{k_j}).$$

An analogous definition holds if we replace N by $\mathbb{Z}$, i.e., if X consists of bi-infinite sequences.

One can see that finite unions of cylinders form an algebra of subsets of X. By Theorem 0.1.5 it can be uniquely extended to a measure on $\mathcal{A}$, the smallest σ-algebra containing all cylinders.

It is often necessary to approximate measurable sets by sets of some sub-class (e.g., an algebra) of the given σ-algebra :

Theorem 0.1.8. *Let $(X, \mathcal{A}, \mu)$ be a probability space, and let $\mathcal{A}_0$ be an algebra of subsets of X generating $\mathcal{A}$. Then, for each $\varepsilon > 0$ and each $A \in \mathcal{A}$ there is some $A_0 \in \mathcal{A}_0$ such that $\mu(A \triangle A_0) < \varepsilon$. Here, $E \triangle F := (E \setminus F) \cup (F \setminus E)$ denotes the* symmetric difference *of E and F.*

0.1.2 The monotone class theorem

Definition 0.1.9. As sequence of sets $\{A_k\}$ is called *increasing* (resp. *decreasing*) if $A_k \subseteq A_{k+1}$ (resp. $A_k \supseteq A_{k+1}$) for all k.
The notation $A_k \uparrow A$ (resp. $A_k \downarrow A$) means that $\{A_k\}$ is an increasing (resp. decreasing) sequence of sets with $\bigcup_k A_k = A$ (resp. $\bigcap_k A_k = A$).

Definition 0.1.10 (Monotone class)**.** Let X be a set. A collection $\mathcal{M}$ of subsets of X is a *monotone class* if whenever $A_k \in \mathcal{M}$ and $A_k \uparrow A$, then $A \in \mathcal{M}$.

Theorem 0.1.11 (Monotone Class Theorem)**.** *A monotone class which contains an algebra, also contains the σ-algebra generated by this algebra.*

Thus, if we show that sets with a certain property form a monotone class, and this class contains an algebra $\mathcal{A}$ of sets, then it contains $\sigma(\mathcal{A})$. For instance, if we can show that two measures μ, ν coincide on an algebra, they coincide on the whole σ-algebra generated by it. This holds true because $\{\mu = \nu\}$ is a monotone class.

0.2 Function spaces

In the following let $k \in \mathrm{N}$ and $0 \le \delta 1$.

Definition 0.2.1 (Functions with Hölder continuous derivatives)**.** a We define $C^{k,\delta}$ as the Frechet space of functions $f:\mathbb{R}^d\to\mathbb{R}^d$ which are k times continuously differentiable, and whose k-th derivative is locally δ-Hölder continuous, when $1>\delta>0$, and locally Lipschitz continuous, when $\delta=1$, with seminorms

$$\|f\|_{k,0;K} := \sum_{0\le|\alpha|\le k} \sup_{x\in K} |\mathrm{D}^\alpha f(x)|\,,$$
$$\|f\|_{k,\delta;K} := \|f\|_{k,0;K} + \sum_{|\alpha|=k} \sup_{x\neq y\in K} \frac{|\mathrm{D}^\alpha f(x)-\mathrm{D}^\alpha f(y)|}{|x-y|^\delta},\ 0<\delta\le 1,$$

where $K\subset\mathbb{R}^d$ are compact convex subsets.

b We define $C_b^{k,\delta}\subset C^{k,\delta}$ as the Banach space of functions $f:\mathbb{R}^d\to\mathbb{R}^d$ with finite norm

$$\|f\|_{k,0} := \sup_{x\in\mathbb{R}^d}\frac{|f(x)|}{1+|x|} + \sum_{1\le|\alpha|\le k} \sup_{x\in\mathbb{R}^d} |\mathrm{D}^\alpha f(x)|\,,$$
$$\|f\|_{k,\delta} := \|f\|_{k,0} + \sum_{|\alpha|=k} \sup_{x\neq y} \frac{|\mathrm{D}^\alpha f(x)-\mathrm{D}^\alpha f(y)|}{|x-y|^\delta},\ 0<\delta\le 1.$$

For functions that are, in addition, time-dependent, we have the following important notions.

Definition 0.2.2. (a) We define $L_{loc}(\mathbb{R},C^{k,\delta})$ as the set of measurable functions $f:\mathbb{R}\times\mathbb{R}^d\to\mathbb{R}^d$, for which

- $f(t,\cdot)\in C^{k,\delta}$ for every $t\in\mathbb{R}$.
- for every compact set $K\subset\mathbb{R}^d$ and every bounded interval $[a,b]\in\mathbb{R}$

$$\int_a^b \|f(t,\dot{)}\|_{k,\delta;K}\mathrm{d}t<\infty, \tag{0.2.1}$$

defining a family of seminorms that makes $L_{loc}(\mathbb{R},C^{k,\delta})$ a Frechet space. The Frechet space $L_{loc}(\mathbb{R},C_b^{k,\delta})$ is defined analagously, adapting (0.2.1) to the full norm.

(b) If $(t,x)\mapsto f(t,x)$ is continuous, we say that $f\in C^{0;k,\delta}$ if

- for each $t\in\mathbb{R}$, $f(t,\cdot)\in C^{k,\delta}$,
- for $k\ge 1$, the derivatives $\mathrm{D}_x^\alpha f(t,x)$ are continuous with respect to (t,x) for all $1\le|\alpha|\le k$,
- for $\delta>0$, the derivatives with $|\alpha|=k$ are locally δ-Hölder continuous with respect to x.

We say that $f\in C_b^{0;k,\delta}$ if $f\in C^{0;k,\delta}$ and for each $t\in\mathbb{R}$, $f(t,\cdot)\in C_b^{k,\delta}$.

0.3 Markov processes and stationary measures

Definition 0.3.1 (Markov process)**.** Let $X, \mathcal{B}$ be a measurable space and $\mathbb{T} \in \{\mathbb{R}_0^+, \mathbb{Z}_0^+\}$. A *Markov process* is defined as a collection of the following objects:

- a measurable space $(\Omega, \mathcal{F})$ with a filtration $\{\mathcal{F}_t, t \in \mathbb{T}\}$
- a family of probability measures $(\mathbb{P}_x)_{x \in X}$ on $(\Omega, \mathcal{F})$ such that the mapping $x \mapsto \mathbb{P}_x(A)$ is measurable for an $A \in \mathcal{F}$
- an X-valued random process $(X_t)_{t \in \mathbb{T}}$ adapted to the filtration $\mathcal{F}_t$ and satisfying the following for any $x \in X, B \in \mathcal{B}$, and $t, s \in \mathbb{T}$:

$$\mathbb{P}_x(\{X_0 = x\}) = 1, \tag{0.3.1}$$

$$\mathbb{P}_x(\{X_{t+s} \in B | \mathcal{F}_s\}) = \hat{P}_t(X_s, B), \quad \text{for } \mathbb{P}_x\text{-almost all } \omega \in \Omega, \tag{0.3.2}$$

where $\hat{P}_t$ is the transition function

$$\hat{P}_t(x, B) := \mathbb{P}_x\{X_t \in B\}, \ x \in X, \ B \in \mathcal{B}. \tag{0.3.3}$$

For any probability measure μ on X, we will write

$$\mathbb{P}\mu = \int_X \mathbb{P}_x \mu(\mathrm{d}x), \ A \in \mathcal{F},$$

and denote the corresponding expectation by $\mathbb{E}_\mu$. It is a famous exercise to check that, for a Markov process $(X_t, \mathbb{P}_x)$ we have for all measurable and bounded $f : X \to \mathbb{R}$ and probability measures μ that

$$\mathbb{E}_\mu[f(X_{t+s}) | \mathcal{F}_s] = \mathbb{E}_{X_s} f(X_t), \ \mathbb{P}_\mu\text{-almost surely.} \tag{0.3.4}$$

To each Markov process, there correspond two families of linear operators, called *Markov semigroups*. They are operating in the space of bounded measurable functions L^∞ and of probability measures $\mathcal{P}(X)$ respectively and defined in terms of the Markov transition function as follows:

$$P_t : L^\infty(X) \to L^\infty(X), \quad P_t f(x) = \int_X \hat{P}_t(x, \mathrm{d}z) f(z), \tag{0.3.5}$$

$$P_t^* : \mathcal{P}(X) \to \mathcal{P}(X), \quad P_t^* \mu(B) = \int_X \hat{P}_t(x, B) \mu(\mathrm{d}x). \tag{0.3.6}$$

It is a straight-forward exercise to observe that the semigroup (P_t) corresponding with the Markov process $(X_t, \mathbb{P}_x)$ can also be written as

$$P_t f(x) = \mathbb{E}_x[f(X_t)] \tag{0.3.7}$$

Definition 0.3.2. A measure $\mu \in \mathcal{P}(X)$ is said to be stationary for the Markov process $(X_t, \mathbb{P}_x)$ if $P_t^* \mu = \mu$ for all $t \in \mathbb{T}$.

Chapter 1

Random dynamical systems and their generators

1.1 Basic definitions

Firstly, we define what we mean by a random dynamical system throughout this lecture. We will consider systems in discrete and continuous time, one-and two-sided. Hence, in the following we will always assume that the index set $\mathbb{T}$ satisfies

$$\mathbb{T} \in \left\{\mathbb{R}, \mathbb{R}_0^+, \mathbb{Z}, \mathbb{Z}_0^+\right\}.$$

A random dynamical system on a measurable space $(X, \mathcal{B})$ consists of

(i) a model of the noise on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$, formalised as a measurable flow $(\theta_t)_{t\in\mathbb{T}}$ of $\mathbb{P}$-preserving transformations $\theta_t : \Omega \to \Omega$,

(ii) a model of the dynamics on X perturbed by noise formalised as a *cocycle* φ over θ.

In technical detail, the definition of a random dynamical system is given as follows:

Definition 1.1.1 (Random dynamical system)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $(X, \mathcal{B})$ be a measurable space.

1. A random dynamical system (RDS) is a pair of mappings (θ, φ) such that the following holds:

   - The $(\mathcal{B}(\mathbb{T}) \otimes \mathcal{F},\ \mathcal{F})$-measurable mapping $\theta : \mathbb{T} \times \Omega \to \Omega$, $(t, \omega) \mapsto \theta_t\omega$, is a *metric dynamical system*, i.e.
     (i) $\theta_0 = \mathrm{id}$ and $\theta_{t+s} = \theta_t \circ \theta_s$ for $t, s \in \mathbb{T}$,
     (ii) $\mathbb{P}(A) = \mathbb{P}(\theta_t^{-1}A)$ for all $A \in \mathcal{F}$ and $t \in \mathbb{T}$.
   - The $(\mathcal{B}(\mathbb{T})\otimes\mathcal{F}\otimes\mathcal{B},\ \mathcal{B})$-measurable mapping $\varphi : \mathbb{T}\times\Omega\times X \to X$, $(t, \omega, x) \mapsto \varphi(t, \omega, x)$, is a cocycle over θ, i.e., $\varphi(0, \omega, \cdot) = \mathrm{id}$ and

$$\varphi(t + s, \omega, \cdot) = \varphi(t, \theta_s\omega, \varphi(s, \omega, \cdot)) \quad \text{for all } \omega \in \Omega \text{ and } t, s \in \mathbb{T}. \tag{1.1.1}$$

2. If X is a topological space with $\mathcal{B} = \mathcal{B}(X)$ its Borel σ-algebra, and

$$(t, x) \mapsto \varphi(t, \omega, x)$$

is continuous for every $\omega \in \Omega$, the random dynamical system (θ, φ) is called *continuous*.

3. If X is additionally a smooth, i.e. C^∞, d-dimensional manifold (e.g. $\mathbb{R}^d$), and for each $(t, \omega) \in \mathbb{T} \times \Omega$ the mapping

$$\varphi(t, \omega) := \varphi(t, \omega, \cdot) : X \to X, \;\; x \mapsto \varphi(t, \omega, x)$$

is C^k, i.e. k-times differentiable in x and the derivatives are continuous in (t, x), the random dynamical system (θ, φ) is called C^k.

We still speak of a random dynamical system, if its cocycle is only defined in forward time, i.e., if the mapping φ is only defined on $\mathbb{R}_0^+ \times \Omega \times X$ or $\mathbb{Z}_0^+ \times \Omega \times X$, while the underlying metric dynamical system is defined in forward and backward time, i.e., the mappings θ_t are defined for all $t \in \mathbb{R}$ or $t \in \mathbb{Z}$ respectively. We will make it noticeable whenever this is the case.

Remark 1.1.2. In the following, the metric dynamical system $(\theta_t)_{t \in \mathbb{T}}$ is often even ergodic, i.e. any $A \in \mathcal{F}$ with $\theta_t^{-1} A = A$ for all $t \in \mathbb{T}$ satisfies $\mathbb{P}(A) \in \{0, 1\}$.

Remark 1.1.3. Further, note that the trajectories of the RDS might explode in finite time. In this case one can consider it as a *local* random dynamical system (as opposed to the *global* random dynamical system from Definition 1.1.1) being defined only for times bounded by some random explosion times $\tau^-(\omega, x)$ and $\tau^+(\omega, x)$. We will consider local RDS in more detail in the context of Chapter 5.

We state our first theorem on two-sided random dynamical systems.

Theorem 1.1.4. *Consider an RDS (θ, φ) on a measurable space $(X, \mathcal{B})$ and two-sided time set $\mathbb{T}$, i.e., $\mathbb{T} = \mathbb{R}$ or $\mathbb{T} = \mathbb{Z}$.*

(a) For all $(t, \omega) \in \mathbb{T} \times \Omega$, the cocycle $\varphi(t, \omega)$ is a bimeasurable bijection of $(X, \mathcal{B})$ and,

$$\varphi(t, \omega)^{-1} = \varphi(-t, \theta_t \omega) \;\; \textit{for all} \;\; (t, \omega) \in \mathbb{T} \times \Omega,$$

or, equivalently,

$$\varphi(-t, \omega) = \varphi(t, \theta_{-t} \omega)^{-1} \;\; \textit{for all} \;\; (t, \omega) \in \mathbb{T} \times \Omega,$$

Furthermore, the mapping

$$(t, \omega, x) \mapsto \varphi(t, \omega)^{-1} x$$

is measurable.

1. *If X is a topological space and the RDS is continuous, then for all $(t, \omega) \in \mathbb{T} \times \Omega$ we have that $\varphi(t, \omega) : X \to X$ is a homeomorphism. If*

(a) $\mathbb{T} = \mathbb{Z}$, *or*

(b) $\mathbb{T} = \mathbb{R}$, *and* X *is a compact Hausdorff space,*

then additionally $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ *is continuous for all* $\omega \in \Omega$.

2. *If* X *is a smooth manifold and the RDS is* C^k, *then for all* $(t, \omega) \in \mathbb{T} \times \Omega$ *we have that* $\varphi(t, \omega) : X \to X$ *is a diffeomorphism. Moreover,* $(t, x) \mapsto \varphi(t, \omega)^{-1}x$ *is* C^k *with respect to* x *for all* $\omega \in \Omega$.

Proof. See Exercise sheet 1. □

Before we address the question of how such random dynamical systems are generated, we introduce a distinction that will be highly relevant when we discuss random dynamical systems in the context of stochastic differential equations. Recall the cocycle property (1.1.1), which in this form is called the *perfect* cocycle property. If equation (1.1.1) holds for fixed $s \in \mathbb{T}$ and all $t \in \mathbb{T}$, $\mathbb{P}$-a.s., where the expectional set N_s with $\mathbb{P}(N_s) = 0$ may depend on s, we call φ a *crude* cocycle. If equation (1.1.1) holds for fixed $s, t \in \mathbb{T}$, $\mathbb{P}$-a.s., where the expectional set $N_{s,t}$ with $\mathbb{P}(N_{s,t}) = 0$ may depend on s, t, we call φ a *very crude* cocycle. The perfection of a very crude cocycle is easy to observe in discrete time but will require some work in continuous time:

Theorem 1.1.5 (Perfection for discrete time)**.** *Let* φ *be a very crude cocycle over* θ *with discrete time* $\mathbb{T}$. *Then there exists a cocycle* ψ *over* θ *which is perfect and indistinguishable from* φ, *i.e., there exists a set* $N \in \mathcal{F}$ *with* $\mathbb{P}(N) = 0$ *and*

$$\{\omega : \psi(t, \omega) \neq \varphi(t, \omega) \text{ for some } t \in \mathbb{T}\} \subset N.$$

Proof. See Exercise sheet 1. □

1.2 Random dynamical systems from products of random mappings

In this section, we focus on random dynamical systems in discrete time $\mathbb{T} \in \{\mathbb{Z}, \mathbb{Z}_0^+\}$. Since, typically, the family of measure-preserving transformations $(\theta_n)_{n\in\mathbb{T}}$ consists of iterations of a map $\theta : \Omega \to \Omega$, we adopt the notation $(\theta^n)_{n\in\mathbb{T}}$ for this section.

Firstly, we make the following observation:

Proposition 1.2.1. *Let* (θ, φ) *be an RDS on* X *with time* $\mathbb{T} \in \{\mathbb{Z}_0^+, \mathbb{Z}\}$.

1. *If* $\mathbb{T} = \mathbb{Z}_0^+$, *we introduce the* time-one mapping

$$\psi(\omega) := \varphi(1, \omega) : X \to X, \tag{1.2.1}$$

and obtain

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \mathrm{id}, & n = 0. \end{cases} \tag{1.2.2}$$

The RDS is measurable if and only if $(\omega, x) \mapsto \psi(\omega)x$ *is measurable. It is continuous/*C^k *if and only if* $x \mapsto \psi(\omega)x$ *is continuous/*C^k*. Conversely, given a family of mappings* $\psi(\omega) : X \to X$ *such that* $(\omega, x) \mapsto \psi(\omega)x$ *is measurable/continuous/*C^k*, then* φ *defined by* (1.2.2) *is the cocycle of a measurable/continuous/*C^k *RDS. We say that* φ *is generated by* ψ*.*

2. *If* $\mathbb{T} = \mathbb{Z}$*, we additionally have the* time-minus-one mapping

$$\varphi(-1, \omega) = \varphi(1, \theta^{-1}\omega)^{-1} = \psi(\theta^{-1}\omega)^{-1} \tag{1.2.3}$$

such that $\psi(\omega) : X \to X$ *is invertible and we obtain*

$$\varphi(n, \omega) = \begin{cases} \psi(\theta^{n-1}\omega) \circ \cdots \circ \psi(\omega), & n \geq 1, \\ \mathrm{id}, & n = 0, \\ \psi(\theta^{n}\omega)^{-1} \circ \cdots \circ \psi(\theta^{-1}\omega)^{-1}, & n \leq -1, \end{cases} \tag{1.2.4}$$

The RDS is measurable if and only if

$$(\omega, x) \mapsto \psi(\omega)x \quad \textit{and} \quad (\omega, x) \mapsto \psi(\omega)^{-1}x \tag{1.2.5}$$

are measurable. It is continuous/C^k *if and only if* $x \mapsto \psi(\omega)x$ *is a homeomorphism/diffeomorphism of order* k*. Conversely, given a family of invertible mappings* $\psi(\omega) : X \to X$ *such that the mappings* (1.2.5) *are measurable/continuous/*C^k*, then* φ *defined by* (1.2.4) *is the cocycle of a measurable/continuous/*C^k *RDS.*

Proof. Straight-forward application of the cocycle property (1.1.1). □

We can put on record: every one-sided (two-sided) discrete time RDS has the form (1.2.2) ((1.2.4)), also called *product of random mappings* or *iterated function system.* Note that we can write the discrete time cocycle $\varphi(n, \omega, x)$ as the solutions of an initial value problem for a random difference equation

$$x_{n+1} = \psi(\theta^n\omega)x_n, \ n \in \mathbb{T}, \ x_0 = x \in X. \tag{1.2.6}$$

Consider the following examples:

Example 1.2.2. *1.* Linear random dynamical system as product of random matrices*: If* $X = \mathbb{R}^d$ *and the RDS is linear, we can write for* $n \geq 1$

$$\varphi(n, \omega) = A_{n-1}(\omega) \cdots A_0(\omega), \ A_k(\omega) = A(\theta^k\omega),$$

where $A : \Omega \to \mathbb{R}^{d\times d}$ *is measurable. Two-sided linear RDS correspond with invertible measurable families of matrices, giving in addition for* $n \leq -1$

$$\varphi(n, \omega) = A_n(\omega)^{-1} \cdots A_{-1}(\omega)^{-1}, \ A_k(\omega) = A(\theta^k\omega).$$

2. Barnsley's chaos game*: Note that one can approximate a Cantor set by randomly switching between the maps*

$$T_0(x) = \frac{x}{2}, \qquad T_1(x) = \frac{1+x}{2}$$

on $X = [0,1]$. Such a switching between random maps can be formalized as an RDS by considering the finite set $\Delta = \{0,1\}$ and the (topological) space of sequences

$$\Omega \equiv \Delta^{\mathbb{N}} := \{\omega = (\omega_n)_{n=0}^{\infty} \mid \omega_n \in \Delta\}.$$

Recall that a cylinder set is of the form

$$C_{i_0,i_1,\ldots,i_n} = \{\omega \in \Omega \mid \omega_k = i_k,\ k = 0,1,\ldots,n\},$$

for some $n \in \mathbb{N}$. Having the two probabilities $1 > p_1 = 1 - p_0 > 0$, we endow the measurable space $(\Omega, \mathcal{B}(\Omega))$ with the infinite product measure $\mathbb{P}$, defined uniquely by its action on cylinder sets as

$$\mathbb{P}(C_{i_0,i_1,\ldots,i_n}) = p_{i_0} \cdots p_{i_n}.$$

The metric dynamical system is given by iterations of the shift map $\theta : \Omega \to \Omega$ defined as

$$\theta(\omega_n)_{n=0}^{\infty} = (\omega_{n+1})_{n=0}^{\infty}.$$

The evolution of the system through time is given by applying the map T_0 or T_1 with probabilities p_0 or p_1, respectively, and this is expressed by the cocycle $\varphi : \mathbb{Z}_0^+ \times X \times \Omega \to X$ as

$$\varphi(0,\omega,x) = x, \qquad \varphi(n,\omega,x) = T_{i_{n-1}} \circ \cdots T_{i_0}(x),$$

where $\omega = (i_k)_{k=0}^{\infty}$. (See also Exercise sheet 1.)

For discrete-time random dynamical systems with independent increments, we can prove the following relation to Markov chains:

Theorem 1.2.3. *Let φ be a measurable cocycle over θ with time $\mathbb{T} = \mathbb{Z}_0^+$, generated by $\psi(\omega)$ such that the sequence $\psi(\theta^n \cdot)$ is identically and independently distributed. Then, given any random variable x_0, the orbit (x_n^x) given by*

$$x_{n+1} = \psi(\theta^n \omega) x_n,\ x_0 = x \in X,$$

is a time-homogeneous Markov chain on X with transition probability

$$P(x,B) = \mathbb{P}\{\omega : \psi(\omega)x \in B\} \quad \text{for all } B \in \mathcal{B}. \tag{1.2.7}$$

Proof. Firstly, note that $P(x,B)$ as defined in (1.2.7) is, indeed, a Markov kernel: $\mathbb{P}(x,\cdot)$ is a probability measure on $(X,\mathcal{B})$ by definition. Furthermore, we observe that $\mathbb{P}(\cdot,B)$ is a measurable map for any $B \in \mathcal{B}$, as follows: Introducing $\Psi : (\omega,x) \mapsto \psi(\omega)x$ and writing

$A_x = \{\omega \in \Omega : (\omega, x) \in A\}$ for $A \in \mathcal{F} \otimes \mathcal{B}$, we have

$$P(x, B) = \mathbb{P}(A_x), \ A = \Psi^{-1} B \in \mathcal{F} \otimes \mathcal{B}.$$

We observe by the monotone class theorem that

$$\mathcal{A} = \{A \in \mathcal{F} \otimes \mathcal{B} : \mathbb{P}(A_x) \text{ is measurable in } x\}$$

is a σ-algebra and, hence, $\mathcal{A} = \mathcal{F} \otimes \mathcal{B}$.

Let us denote $\mathcal{F}_n = \sigma(x_0^x, \dots, x_n^x; x \in X)$. Since $1_B(\psi(\theta^n \cdot)x)$ is independent from $\mathcal{F}_n$ for each $x \in X$ and $B \in \mathcal{B}$, we can deduce by the well-know properties of conditional expectations that

$$\mathbb{P}(x_{n+1}^x \in B | \mathcal{F}_n) = \mathbb{E}[1_B(\psi(\theta^n \omega) x_n^x) | \mathcal{F}_n] = \mathbb{E}[1_B(\psi(\theta^n \omega) x_n^x) | x_n^x] = \mathbb{P}(x_{n+1}^x \in B | x_n^x).$$

This shows the Markov property. Moreover, we obtain the time-homogeneity

$$\mathbb{P}(x_{n+1}^x \in B | x_n^x = y) = \mathbb{P}(\omega : \psi(\theta^n \omega) y \in B) = \mathbb{P}(\omega : \psi(\omega) y \in B) = P(y, B),$$

having used the θ^n-invariance of $\mathbb{P}$ for all $n \geq 1$. □

Remark 1.2.4. The reverse direction, i.e., the consruction of a discrete-time random dynamical system as a composition of independent random maps from a Markov chain with given transition probabilities, is also possible (see [19, Theorem 1.1]), but, in general, uniqueness cannot be guaranteed. This has to do with the RDS perspective of providing a description of the n-point motion, i.e., tracking trajectories with different initial conditions but driven by the same noise, whereas the Markov chain only describes the 1-point motion. We will discuss this distinction in more detail later on.

[**End of Lecture I, 13.04.**]

1.3 Random dynamical systems from random differential equations

In the following, let time be continuous and two-sided, i.e. $\mathbb{T} = \mathbb{R}$, and the state space $X = \mathbb{R}^d$ (we could also consider manifolds but concentrate on the Euclidean case for simplicity).

We consider *random differential equations* (RDEs) of the form

$$\dot{x}_t = f(\theta_t \omega, x_t), \tag{1.3.1}$$

for some measurable map $f : \Omega \times \mathbb{R}^d \to \mathbb{R}^d$, where θ is a metric dynamical system as before. We say that the map $t \mapsto \varphi(t, \omega, x)$ solves the RDE (1.3.1) (or the RDE (1.3.1) generates φ), if we have

$$\varphi(t,\omega,x) = x + \int_0^t f(\theta_s\omega, \varphi(s,\omega,x))\,\mathrm{d}s. \tag{1.3.2}$$

We can state the following theorem:

Theorem 1.3.1. *Consider the RDE* (1.3.1) *and write* $f_\omega(t,x) = f(\theta_t\omega, x)$. *Then the following holds:*

(a) If $f_\omega \in L_{loc}(\mathbb{R}, C_b^{0,1})$ *for all* $\omega \in \Omega$*, then equation* (1.3.1) *uniquely generates a continuous RDS* φ *over* θ*. If* $f_\omega \in C_b^{0;0,1}$ *for all* $\omega \in \Omega$*, then* φ *is differentiable with respect to* t*.*

(b) If $f_\omega \in L_{loc}(\mathbb{R}, C_b^{k,0})$ *for all* $\omega \in \Omega$ *and some* $k \geq 1$*, then equation* (1.3.1) *uniquely generates a* C^k *RDS* φ *over* θ*. If* $f_\omega \in C_b^{0;k,0}$ *for all* $\omega \in \Omega$*, then* φ *is differentiable with respect to* t*.*

(c) In case (b), consider the Jacobian of $\varphi(t,\omega)$ *at* $x \in \mathbb{R}^d$*,*

$$\mathrm{D}_x\varphi(t,\omega,x) := \left(\frac{\partial_{x_i}\varphi(t,\omega,x)}{\partial x_j}\right)_{1\leq i,j\leq d}.$$

Then $(\varphi, \mathrm{D}_x\varphi)$ *is a* C^{k-1} *RDS on* $\mathbb{R}^d \times \mathbb{R}^d$ *over* θ *uniquely generated by the RDE on the tangent bundle*

$$(\dot{x}_t, \dot{v}_t) = (f(\theta_t\omega, x_t), \mathrm{D}_x f(\theta_t\omega, x_t)v_t).$$

The Jacobian $\mathrm{D}_x\varphi$ *uniquely solves the* variational equation

$$\mathrm{D}_x\varphi(t,\omega,x) = \mathrm{Id} + \int_0^t \mathrm{D}_x f(\theta_s\omega, \varphi(s,\omega,x))\mathrm{D}_x\varphi(s,\omega,x)\mathrm{d}s,$$

and is a matrix cocycle over the skew product $\Theta_t(\omega,x) := (\theta_t\omega, \varphi(t,\omega,x))$*.*

The determinant $\det \mathrm{D}_x\varphi(t,\omega,x)$ *satisfies* Liouville's equation

$$\det \mathrm{D}_x\varphi(t,\omega,x) = \exp \int_0^t (\mathrm{trace}\, \mathrm{D}_x f)(\theta_s\omega, \varphi(s,\omega,x))\mathrm{d}s,$$

and is a scalar cocycle over Θ*.*

Proof. The proofs of (a) and (b) can be directly obtained from classical ODE theory (see e.g. [1]), now applied for each $\omega \in \Omega$. For statement (c), see exercise sheet 2. □

We would like to give a stationary characterization of sufficient conditions for finding random dynamical systems generated by random differential equation. A crucial step towards this direction is the following observation.

Lemma 1.3.2. *Let* $(\Omega, \mathcal{F}, \mathbb{P}, (\theta_t)_{t\in\mathbb{R}})$ *be a metric dynamical system and* $g \in L^1_{\mathbb{P}}(\Omega)$*. Then the measurable stationary process* $t \mapsto g(\theta_t\omega)$ *is locally integrable on an invariant set* $\tilde{\Omega} \subset \Omega$ *with* $\mathbb{P}(\tilde{\Omega}) = 1$*, and for all* $a, b \in \mathbb{R}$

$$\int_a^b g(\theta_t\cdot)\mathrm{d}t \in L^1_{\mathbb{P}}(\Omega).$$

Proof. The set

$$\tilde{\Omega} := \{\omega \in \Omega \,:\, t \mapsto g(\theta_t \omega) \text{ is locally integrable}\}$$

is clearly measurable and θ_t-invariant (check!). We write $m_g := \mathbb{E}\,|g(\theta_t \cdot)| < \infty$ and for all $a \leq b \in \mathbb{R}$, using Fubini,

$$\mathbb{E} \int_a^b |g(\theta_t \cdot)| \,\mathrm{d}t = \int_a^b \mathbb{E}\,|g(\theta_t \cdot)| \,\mathrm{d}t = m(b-a) < \infty,$$

such that, indeed, $\mathbb{P}(\tilde{\Omega}) = 1$. □

We can now directly deduce the following theorem:

Theorem 1.3.3. *(a) If $f \in L^1_{\mathbb{P}}(\Omega, C_b^{0,1})$ (or if $f \in L^1_{\mathbb{P}}(\Omega, C_b^{k,0})$, then $f_\omega \in L^1_{loc}(\mathbb{R}, C_b^{0,1})$ (or $f_\omega \in L^1_{loc}(\mathbb{R}, C_b^{k,0})$ on an invariant set of full measure, and the random differential equation* (1.3.1) *uniquely generates a continous (or C^k) RDS.*

(b) This is, more generally, also the case, if $f_\omega \in L_{loc}(\mathbb{R}, C^{0,1})$ (or if $f_\omega \in L^1_{loc}(\mathbb{R}, C^{k,0})$, and

$$\|f(\omega, x)\| \leq \alpha(\omega)\|x\| + \beta(\omega), \quad \alpha, \beta \in L^1_{\mathbb{P}}(\Omega). \tag{1.3.3}$$

Proof. Part (a) follows from the definition of the spaces $L_{loc}(\mathbb{R}, C^{0,1}), L_{loc}(\mathbb{R}, C_b^{0,1})$ and applying Lemma 1.3.2 to $g(\omega) = \|f(\omega, \cdot)\|_{0,1;K}$, and the other norms similarly. Part (b) follows from the definition of these norms (see Section 0.2). □

1.4 Random dynamical systems from stochastic differential equations

1.4.1 Short overview on SDEs

For a more comprehensive summary concerning stochastic differential equation (SDEs), see for example [26]. The following is just a short (not completely rigorous) recap of some important notions.

Firstly, recall the following definition:

Definition 1.4.1 (Brownian motion)**.** (a) A real-valued stochastic process $W_t(\omega)$ defined on $\mathbb{R}_0^+ \times \Omega$ is a Brownian motion if

- $W_0(\omega) = 0$ almost surely,
- $W_t(\omega)$ is almost surely continuous in t
- For every $t, s \geq 0$, the increments $\Delta W_s(\omega) = W_{t+s}(\omega) - W_t(\omega)$ are independent from $W_u, u \leq t$, and zero Gaussian random variables with variables with variance

$$\mathbb{E}\,|\Delta W_s|^2 = s.$$

(b) If $W_t^1, W_t^2, \ldots, W_t^d$ are independent Browinan motions, the vector process

$$W_t = (W_t^1, W_t^2, \ldots, W_t^d)^\top$$

is called *d-dimensional Brownian motion.*

Brownian paths are *rough*, i.e. not very regular in the sense that they are not differentiable at any t with probability one but only (locally) δ-Hölder continuous for $0 < \delta < \frac{1}{2}$. Hence, defining integration with respect to Brownian motion is not straight-forward. Here, we just briefly sketch the meaning of two forms of stochastic differential equations.

Firstly, we consider stochastic differential equations (SDEs) on $\mathbb{R}^d$ of *Itô type*

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t)\,\mathrm{d}W_t\,, \quad X_0 = x \in \mathbb{R}^d\,, \tag{1.4.1}$$

written in integral form as

$$X_t = x + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s)\,\mathrm{d}W_s\,, \quad x \in \mathbb{R}^d\,, \tag{1.4.2}$$

where

- W_t denotes m-dimensional Brownian motion, $m \leq d$,
- $f : \mathbb{R}^d \to \mathbb{R}^d$ and $g : \mathbb{R}^d \to \mathbb{R}^{d\times m}$ are globally Lipschitz continuous and of at most linear growth, i.e. $f, g_j \in C_b^{0,1}(\mathbb{R}^d)$, where g_j denote the columns of g.

It can be shown [26, Section 3.3] that for any sequence of partitions of a time interval $0 \leq t_0^k < t_1^k < \cdots < t_n^k = t$ such that $\max_i(t_i^k - t_{i-1}^k) \to 0$, we obtain a limit in probability

$$\int_0^t g(X_s)\,\mathrm{d}W_s = \lim_{\max_i(t_i - t_{i-1})\to 0} \sum_{i=1}^n g(X_{\mathbf{t_{i-1}}})[W_{t_i} - W_{t_{i-1}}],$$

which is called the *Itô integral.* Under the above conditions on the coefficients f and g, one can then show [26, Theorem 4.1.1] that there is an almost surely continuous process X_t adapted to the filtration $\mathcal{F}_t$ generated by the Brownian motion W_t (see below) that solves equation (1.4.2) for almost all $\omega \in \Omega$.

Similarly, we consider SDEs on $\mathbb{R}^d$ of *Stratonovich type*

$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + g(X_t) \circ \mathrm{d}W_t\,, \quad X_0 = x \in \mathbb{R}^d\,, \tag{1.4.3}$$

written in integral form as

$$X_t = x + \int_0^t f(X_s)\,\mathrm{d}s + \int_0^t g(X_s) \circ \mathrm{d}W_s\,, \quad x \in \mathbb{R}^d\,, \tag{1.4.4}$$

where the stochastic integral is given as the limit in probability

$$\int_0^t g(X_s) \circ \mathrm{d}W_s = \lim_{\max_i (t_i - t_{i-1}) \to 0} \sum_{i=1}^n g\left(X_{\frac{\mathbf{t_i + t_{i-1}}}{\mathbf{2}}} \right) [W_{t_i} - W_{t_{i-1}}].$$

When the diffusion coefficient g is differentiable, we can apply the famous conversion formula between Stratonovich and Itô integrals, also called *Wong-Zakai corrections*:

$$\left(\int_0^t g(X_s) \circ \mathrm{d}W_s \right)_i = \left(\int_0^t g(X_s)\, W_s \right)_i + \frac{1}{2} \int_0^t \sum_{j=1}^d \sum_{k=1}^m g_{jk}(X_s) \frac{\partial g_{ik}}{\partial X_j}(X_s)\, \mathrm{d}s. \tag{1.4.5}$$

Hence, unique solvability of (1.4.4) can then be directly obtained from unique solvability of (1.4.2).

Remark 1.4.2 (Important properties of SDEs)**.** (a) The unique solutions of these SDEs are Markov processes.

(b) The Stratonovich integral satisfies the usual rules of calculus (product rule and chain rule).

(c) The Itô integral satisfies the chain rule under adding an additional term, i.e., Itô's formula for $h \in C^2(\mathbb{R}^d, \mathbb{R})$ and the SDE (1.4.1) reads

$$\mathrm{d}h(X_t) = \nabla h(X_t) \cdot f(X_t)\, \mathrm{d}t + \nabla h(X_t) \cdot (g(X_t)\, \mathrm{d}W_t) + \frac{1}{2} \operatorname{trace}(g^* \operatorname{hess}(h) g)\, \mathrm{d}t. \tag{1.4.6}$$

(d) Itô integrals are martingales.

1.4.2 The RDS framework for SDEs

Before we state a theorem about the generation of random dynamical systems by stochastic differential equations, we introduce the more general notion of random dynamical systems adapted to a suitable filtration and of white noise type. Following [14], we make the following definition:

Definition 1.4.3 (White noise RDS)**.** Let (θ, φ) be a random dynamical system over a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ on a topological space X where φ is defined in forward time and θ is defined in two-sided time. Let $(\mathcal{F}_s^t)_{-\infty \leq s \leq t \leq \infty}$ be a family of sub-σ-algebras of $\mathcal{F}$ such that

(i) $\mathcal{F}_t^u \subset \mathcal{F}_s^v$ for all $s \leq t \leq u \leq v$,

(ii) $\mathcal{F}_s^t$ is independent from $\mathcal{F}_u^v$ for all $s \leq t \leq u \leq v$,

(iii) $\theta_r^{-1}(\mathcal{F}_s^t) = \mathcal{F}_{s+r}^{t+r}$ for all $s \leq t$, $r \in \mathbb{R}$,

(iv) $\varphi(t, \cdot, x)$ is $\mathcal{F}_0^t$-measurable for all $t \geq 0$ and $x \in X$.

Furthermore we denote by $\mathcal{F}_{-\infty}^t$ the smallest $sigma$-algebra containing all $\mathcal{F}_s^t$, $s \leq t$, and by $\mathcal{F}_t^\infty$ the smallest $sigma$-algebra containing all $\mathcal{F}_t^u$, $t \leq u$. Then (θ, φ) is called a *white noise (filtered) random dynamical system.*

Consider an SDE, say of Stratonovich type (1.4.3), that can be solved uniquely. We need the following probabilistic setting to identfy the solution process with a random dynamical system:

Definition and Proposition 1.4.4 (Brownian motion as dynamical system)**.** We introduce the canonical (two-sided) *Wiener space* $(\Omega, \mathcal{F}, \mathbb{P})$ with

1. $\Omega = C_0(\mathbb{R}, \mathbb{R}^d)$, i.e. the space of all continuous functions $\omega : \mathbb{R} \to \mathbb{R}^d$ satisfying that $\omega(0) = 0 \in \mathbb{R}^d$. We endow Ω with the *compact-open* topology given by the complete metric
$$\rho(\omega, \widehat{\omega}) := \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\|\omega - \widehat{\omega}\|_n}{1 + \|\omega - \widehat{\omega}\|_n}, \quad \|\omega - \widehat{\omega}\|_n := \sup_{|t| \leq n} \|\omega(t) - \widehat{\omega}(t)\| .$$

2. $\mathcal{F} = \mathcal{B}(\Omega)$, the Borel σ-algebra on (Ω, ρ).

3. the *Wiener measure* $\mathbb{P}$ on $(\Omega, \mathcal{F})$ such that the d processes $(W_t^1), \dots, (W_t^d)$ defined by $(W_t^1(\omega), \dots, W_t^d(\omega))^\top := \omega(t)$ for $\omega \in \Omega$ are independent one-dimensional Brownian motions, i.e. for all $x \in \mathbb{R}^d$
$$\mathbb{P}(\{\omega \in \Omega : \omega_1(t) \leq x_1, \dots, \omega_d(t) \leq x_d\}) = \frac{1}{(2\pi t)^{d/2}} \int_{-\infty}^{x_1} \cdots \int_{-\infty}^{x_d} e^{-\|y\|^2/2|t|} \mathrm{d}y_1 \cdots \mathrm{d}y_d.$$

4. the sub-σ-algebra $\mathcal{F}_s^t$ as the σ-algebra generated by $\omega(u) - \omega(v)$ for $s \leq v \leq u \leq t$.

The family of shifts $(\theta_t)_{t \in \mathbb{R}}$ on $(\Omega, \mathcal{F}, \mathbb{P})$, given by
$$\theta_t \omega(\cdot) := \omega(t + \cdot) - \omega(t)$$
is measure-preserving and ergodic.

Proof. The proof is very similar to Exercises 1 and 2 on the first question sheet, using the time-one maps $\omega(n), n \in \mathbb{Z}$. Note, in particular, that we can also define the topology and σ-algebra, and thereby the measure $\mathbb{P}$, via cylinder sets in Ω (cf. [26, Section 2.2]). □

[**End of Lecture II, 20.04.**]

1.4.3 Main theorem on generation of RDS from SDEs

In this framework, we can now formulate that main theorem of this section. Note that we can completely formulate it in two-sided time, due to the possibility of using a backward stochastic integral for $t < 0$, but will later often only consider the stochastic integral in classical forward time, in particular in the context of Markov processes:

Theorem 1.4.5 (Cocycle through SDE)**.** *Consider the Stratonovich SDE, where we explcitly right the columns g_j of the diffusion matrix g,*
$$\mathrm{d}X_t = f(X_t)\,\mathrm{d}t + \sum_{j=1}^{m} g_j(X_t) \circ \mathrm{d}W_t^j\,, \quad X_0 = x \in \mathbb{R}^d\,, t \in \mathbb{R}\,, \tag{1.4.7}$$

where, for some $k \in \mathbb{N}$ and $\delta \in (0,1]$, we have $f \in C_b^{k,\delta}$ and $g_j \in C_b^{k+1,\delta}$ for all $j = 1, \dots, m$.

Then there is a unique measurable function $(t, \omega, x) \mapsto \varphi(t, \omega, x)$ such that

- *(θ, φ) is a C^k RDS,*
- *$\varphi(t, \cdot, x)$ is the solution of the SDE* (1.4.7).

Proof. See [3] for details. We give the following sketch of proof:

1. As a first step, one may refer to Kunita [22, Chapter 4.7] to establish the fact that solutions of (1.4.7) can be expressed as a measurable function $(s, t, \omega, x) \mapsto \varphi_{st}(\omega, x)$ such that for all $\omega \in \Omega$

$$(s, t, x) \mapsto \varphi_{st}(\omega, x) \quad \text{is continuous for all} s \leq t \in \mathbb{R},$$

and $\varphi_{st} := \varphi_{st}(\omega, \cdot)$ is a two parameter flow of C^k diffeomorphisms, i.e.,

$$\varphi_{rt}(\omega) \circ \varphi_{sr}(\omega) = \varphi_{st}(\omega), \quad \varphi_{ss}(\omega) = \mathrm{id}\,. \tag{1.4.8}$$

Small exercise: check that the simplest possible SDE $\mathrm{d}X_t = \mathrm{d}W_t$ generates a C^∞ flow $x \mapsto \varphi_{st}(\omega, x) = x + W_t(\omega) - W_s(\omega)$.

2. As a second step, we may observe by the uniqueness of solution that

$$\varphi_{rt}(\omega) = \varphi_{0,t-r}(\theta_r \omega), \qquad \text{for almost all } \omega \in \Omega.$$

Small exercise: again check this for the simplest possible SDE $\mathrm{d}X_t = \mathrm{d}W_t$.

By setting $\varphi(t, \cdot) := \varphi_{0t}(\cdot)$, the two-parameter flow property (1.4.8) becomes (by replacing t, r, s with $t + s, s, 0$) the cocycle property

$$\varphi(t + s, \omega) = \varphi(t, \theta_s \omega) \circ \varphi(s, \omega), \text{ for almost all } \omega \in \Omega \text{ and all } s, t \in \mathbb{R}.$$

The crucial observation is that this only gives the crude cocycle property since the exceptional null set may depend on s.

3. Finally, one needs to prove the perfection of cocycle via an abstract argument, using the notion of *Haar measure.* [3, Section 5].

Please follow the literature if you are intrested in the highly abstract and lengthy proof. □

Remark 1.4.6. We are not always able to work with a global Lipschitz condition which in many interesting dynamical examples is not satisfied. Instead, we sometimes use a transformation into a random differential equation to show that the respective stochastic differential equation, indeed, generates a (global) random dynamical system. For example, consider for $\sigma \geq 0$ an SDE of the form

$$\mathrm{d}X_t = f(X_t)\mathrm{d}t + \sigma \mathrm{d}W_t\,, \tag{1.4.9}$$

where f is continuous and has dissipative properties but is not globally Lipschitz continuous or has linearly bounded growth. Consider, for some $c > 0$, the stationary *Ornstein-Uhlenbeck process*

$$Z^*(\theta_t\omega) = Z^*(\omega) - c\int_0^t Z^*(\theta_s\omega)\,\mathrm{d}s + \omega(t)\,.$$

An RDS solution to (1.4.9), i.e.,

$$\varphi(t,\omega,Z) = Z + \int_0^t f(\varphi(s,\omega,Z))\,\mathrm{d}s + \sigma\omega(t) \quad \text{for all } t \in [0,T]\,.$$

can be found via the conjugated RDE

$$\dot{Y} = g(\theta_t\omega, Y)\,,$$

where $g(\omega,Y) := f\left(T(\omega,Y)\right) + c\sigma Z^*(\omega)$ and $T(\omega,Y) := Y + \sigma Z^*(\omega)$. See Question sheet 2 for more details.

1.5 Invariant measures

1.5.1 Basic definition

Let (θ,φ) be a random dynamical system with the cocycle φ. Then the system generates a skew product flow, i.e. a family of maps $(\Theta_t)_{t\in\mathbb{T}}$ from $\Omega\times X$ to itself such that for all $t\in\mathbb{T}$ and $\omega\in\Omega, x\in X$

$$\Theta_t(\omega,x) = (\theta_t\omega, \varphi(t,\omega,x))\,.$$

The notion of an invariant measure for the random dynamical system is given via the invariance with respect to the skew product flow, see e.g. [2, Definition 1.4.1]. We denote by $T^*\mu$ the push forward of a measure μ by a map T, i.e. $T^*\mu(\cdot) = \mu(T^{-1}(\cdot))$.

Definition 1.5.1 (Invariant measure)**.** A probability measure μ on $(\Omega\times X, \mathcal{F}\otimes\mathcal{B})$ is invariant for the random dynamical system (θ,φ) if

(i) $\Theta_t^*\mu = \mu$ for all $t\in\mathbb{T}$,

(ii) the marginal of μ on Ω is $\mathbb{P}$, i.e. μ can be factorised uniquely into

$$\mu(\mathrm{d}\omega,\mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega),$$

where $\omega\mapsto\mu_\omega$ is a *random measure* (or *disintegration* or *sample measure*) on X, i.e.

- μ_ω is a probability measure on X for $\mathbb{P}$-a.a. $\omega\in\Omega$,
- $\omega\mapsto\mu_\omega(B)$ is measurable for all $B\in\mathcal{B}(X)$.

Recall that we assume the model of the noise to be fixed. Hence, the marginal of μ on the probability space is demanded to be $\mathbb{P}$.

We observe the following for topological spaces X with countably generated $\mathcal{B}(X)$ (e.g. Polish spaces):

Proposition 1.5.2. *A probability measure μ on $(\Omega \times X, \mathcal{F} \otimes \mathcal{B})$ with marginal $\mathbb{P}$ is invariant for the random dynamical system (θ, φ) if and only if the sample measures satisfy*

$$\varphi(t, \omega, \cdot)\mu_\omega = \mu_{\theta_t \omega} \quad \mathbb{P}\text{-a.s. for all } t \in \mathbb{T}\,. \tag{1.5.1}$$

Proof. See Question sheet 3. □

For white noise random dynamical systems (θ, φ) as given by Drfinition 1.4.3, in particular random dynamical systems induced by a stochastic differential equation, there is a one-to-one correspondence between certain invariant random measures and stationary measures of the associated Markov process, first observed in [8]. In the following, we will formulate this a general property of *Markov RDS*.

1.5.2 Markov RDS and correspondence theorem

Let in the the following (θ, φ) be an RDS on a topological space X, where $(\theta_t)_{t \in \mathbb{T}}$ is two-sided in time and the cocycle is defined for one-sided time $t \in \mathbb{T}_+$. For any $u, v \in \mathbb{T}$ with $u < v$, we denote by $\mathcal{F}_u^v \subset \mathcal{F}$ the sub-σ-algebra generated by (the subsets of $\mathcal{F}$ of zero measure) and the random variable $\varphi(t, \theta_s \omega, x)$ for $x \in X$ and $t, s \in \mathbb{T}$ with $u \leq s \leq v$ and $0 < t \leq v - s$.

This means, in particular, that $\varphi(t, \theta_s \omega, x)$ is $\mathcal{F}_s^{s+t}$-measurable. We define the σ-algebras

$$\begin{aligned}
\mathcal{F}_{-\infty}^v &= \sigma(\mathcal{F}_u^v \,:\, u \in \mathbb{T}, u < v), \\
\mathcal{F}_u^\infty &= \sigma(\mathcal{F}_u^v \,:\, v \in \mathbb{T}, u < v), \\
\mathcal{F}_{-\infty}^\infty &= \sigma(\mathcal{F}_u^v \,:\, v, u \in \mathbb{T}, u < v)
\end{aligned}$$

Definition 1.5.3. The RDS (θ, φ) is said to be *Markov* if $\mathcal{F}_{-\infty}^0$ and $\mathcal{F}_0^\infty$ are indepedent, i.e., if past and future are independent.

Fur such random dynamical systems, we can prove the following version of the Markov property:

Proposition 1.5.4. *Let the RDS (θ, φ) be Markov and consider the filtration given by $\mathcal{F}_s = \mathcal{F}_{-\infty}^s$ for all $s \in \mathbb{T}$. Then, for any $\mathcal{F}_{-\infty}^0$-measurable random variable $Y : \Omega \to X$ and any $s, t \in \mathbb{T}^+$, we have*

$$\mathbb{E}[f(\varphi(t+s, \omega, Y(\omega))|\mathcal{F}_s] = \mathbb{E}[f(\varphi(t, \cdot, Z)]|_{Z=\varphi(s,\omega,Y(\omega))} \quad \text{for } \mathbb{P}\text{-almost all } \omega, \tag{1.5.2}$$

for ally bounded and measurable functions $f : X \to \mathbb{R}$.

Proof. Firstly, note that for any finite or infinite numbers $u < v$, we have for all $t \in \mathbb{T}$ that

$$\theta_t^{-1} \mathcal{F}_u^v = \mathcal{F}_{u+t}^{v+t}.$$

We may conclude that for the given Markov RDS, the σ-algebras $\mathcal{F}_{\infty}^{t}$ and $\mathcal{F}_{t}^{\infty}$ are independent for any $t \in \mathbb{T}$.

Now, consider the function $g : \Omega \times X \to \mathbb{R}$ defined as

$$g(\omega, x) = f(\varphi(t, \theta_s \omega, x)),\ \omega \in \Omega,\ x \in X.$$

Then, for any $x \in X$, we have that $g(\cdot, x)$ is $\mathcal{F}_s^{\infty}$-measurable and that $\varphi(s, \cdot, x)$ is $\mathcal{F}_{-\infty}^{s}$-measurable. Due to the independence established above we obtain for almost all $\omega \in \Omega$

$$\mathbb{E}[g(\omega, \varphi(s, \omega, Y(\omega)))|\mathcal{F}_{-\infty}^{s}] = \mathbb{E}[g(\cdot, Z)]|_{Z=\varphi(s,\omega,Y(\omega))}.$$

Using the cocycle property, we observe that

$$g(\omega, \varphi(\omega, s, Y(\omega))) = f(\varphi(t, \theta_s \omega, \varphi(s, \omega, Y(\omega))) = f(\varphi(t + s, \omega, Y(\omega))).$$

Hence, we can coclude that for almost all $\omega \in \Omega$

$$\mathbb{E}[f(\varphi(t + s, \omega, Y(\omega))|\mathcal{F}_s] = \mathbb{E}[f(\varphi(t, \theta_s \cdot, Z)]|_{Z=\varphi(s,\omega,Y(\omega))},$$

which concludes the proof. □

We can now introduce the transition function for the Markov RDS given by

$$\hat{P}_t(x, B) = \mathbb{P}(\{\varphi(t, \cdot, x) \in B\}),\ x \in X,\ B \in \mathcal{B}(X),\ t \in \mathbb{T}_+. \tag{1.5.3}$$

It is now straightforward to construct a Markov process out of the Markov RDS in a canonical way: We denote by $\Omega' = X \times \Omega$ the product space endowed with the product σ-algebra $\mathcal{B}(X) \otimes \mathcal{F}$ and introduce $\mathcal{F}_t' = \mathcal{B}(X) \times \mathcal{F}_t$, where $\mathcal{F}_t = \mathcal{F}_{-\infty}^{t}$ as before. Defining the process $X_t(\omega') = \varphi(t, \omega, x)$ with $\omega' = (x, \omega) \in \Omega'$ and the family of probability measures $\mathbb{P}_x = \delta_x \otimes \mathbb{P}$, we observe that $(X_t, \mathbb{P}_x)$ gives a Markov process with transition function $\hat{P}_t$ (1.5.3). We denote the associated semigroups by P_t and P_t^*, cf. Section 0.3.

In the following, when we consider weak convergence of measures $\mu_k \to \mu$ on a space X, we will always mean that

$$\langle f, \mu_k \rangle := \int_X f(x)\mu_k(\mathrm{d}x) \to \int_X f(x)\mu(\mathrm{d}x) =: \langle f, \mu \rangle$$

for all $f \in C_b(X)$, i.e. all bounded and continuous functions from X to $\mathbb{R}$.

We can now turn to the main result of this section, as announced, using one last definition:

Definition 1.5.5 (Markov measure)**.** An invariant measure μ for an RDS (θ, φ) is said to be *Markov* if its disintegration $\{\mu_\omega\}$ is $\mathcal{F}_{-\infty}^{0}$-measurable, i.e., for any $B \in \mathcal{B}(X)$ the map $\omega \mapsto \mu_\omega(B)$ is $(\mathcal{F}_{-\infty}^{0}, \mathcal{B}(\mathbb{R}))$-measurable.

Theorem 1.5.6 (Correspondence theorem)**.** *Let (θ, φ) be a Markov RDS in a separable completely metrizable toplogical space X (also called Polish space). Then the following assertions hold:*

(a) Let μ be an invariant Markov measure with disintegrations μ_ω. Then $\rho = \mathbb{E}[\mu_\omega]$ is a stationary measure for the associated Markov process, i.e., $P_t^* \rho = \rho$.

Additionally, if μ' is another invariant Markov measure with disintegrations μ'_ω such that $\mathbb{E}[\mu_\omega] = \mathbb{E}[\mu'_\omega]$, then $\mu = \mu'$.

(b) Let ρ be a stationary measure for the associated Markov process. Then for any sequence $t_k \to \infty$, there is a set $\tilde{\Omega} \in \mathcal{F}$ of full measure such that there is a weak limit

$$\mu_\omega = \lim_{k\to\infty} \varphi(t_k, \theta_{-t_k}\omega, \cdot)^*(\rho) \tag{1.5.4}$$

for any $\omega \in \tilde{\Omega}$. If there is another sequence $t'_k \to \infty$, then the corresponding μ'_ω coincide with μ_ω almost surely. Finally, the measure μ defined by its disintegrations μ_ω is an invariant Markov measure, and $\rho = \mathbb{E}[\mu_\omega]$.

Summarizingly, there is a one-to-one correspondence between stationary and invariant Markov measures for the RDS.

[**End of Lecture III, 27.04.**]

Proof. (a) See Question sheet 3, Exercise 2 (a), for the proof that $\rho := \mathbb{E}[\mu_\omega]$ is a stationary measure.

We now prove the second part of (a). We fix an increasing sequence, and, using Proposition 1.5.2, we have almost surely that for all $k \geq 1$

$$\varphi(t_k, \theta_{-t_k}\omega, \cdot)^* \mu_{\theta_{-t_k}\omega} = \mu_\omega.$$

Hence, we obtain for any $f \in C_b(X)$ that

$$\langle f(\varphi(t_k, \theta_{-t_k}\omega, \cdot), \mu_{\theta_{-t_k}\omega}\rangle = \langle f, \mu_\omega\rangle. \tag{1.5.5}$$

Setting $\mathcal{G}_k := \mathcal{F}^0_{-t_k}$ for $k \geq 1$, we observe that $f(\varphi(t_k, \theta_{-t_k}\omega, x)$ is $\mathcal{G}_k$-measurable for all $x \in X$, while $\mu_{\theta_{-t_k}\omega}$ is independent of $\mathcal{G}_k$. Taking conditional expectation with respect to $\mathcal{G}_k$ on both sides of (1.5.5) and writing again $\rho := \mathbb{E}[\mu_\omega]$, we get for almost all $\omega \in \Omega$ that

$$\langle f(\varphi(t_k, \theta_{-t_k}\omega, \cdot), \rho\rangle = \mathbb{E}[\langle f, \mu_\omega\rangle | \mathcal{G}_k]. \tag{1.5.6}$$

Since μ_ω is $\mathcal{F}^0_{-\infty}$-measurable and $\mathcal{F}^0_{-\infty} = \sigma\{\mathcal{G}_k, k \geq 1\}$, we observe that the right-hand side of equation (1.5.6) is a right-closable martingale with respect to the filtration $\{\mathcal{G}_k\}$. Hence, Doob's convergence theorem on right-closed martingale sequences yields

$$\lim_{k\to\infty} \langle f(\varphi(t_k, \theta_{-t_k}\omega, \cdot), \rho\rangle = \langle f, \mu_\omega\rangle$$

for any $f \in C_b(X)$ and almost all $\omega \in \Omega$. It follows that μ_ω is uniquely defined by ρ such that $\mu' = \mu$ for any invariant Markov measure μ' for which $\rho' = \rho$.

(b) Take an arbitrary sequence $t_k \to \infty$ and assume that is is increasing (without loss of generality). Given a function $f \in C_b(X)$, consider the sequence

$$\xi_k(\omega) = \langle f, \varphi(t_k, \theta_{-t_k}\omega, \cdot)^*\rho\rangle.$$

One may observe that $\{\xi_k\}$ is bounded uniformly in k and ω and a martingale with respect to the filtration $\mathcal{G}_k = \mathcal{F}^0_{-t_k}$. Hence, by Doob's martingale convergence theorem, $\xi_k(\omega)$ converges almost surely (see Question Sheet 3, Exercise 2). Therefore, by [21, Theorem 7.5.2] (this is where it is needed that X is a Polish space), there is a random probability measure $\{\mu_\omega\}$ and set of full measure $\tilde{\Omega}$ such that (1.5.4) holds for all $\omega \in \tilde{\Omega}$.

The fact that the limit (1.5.4) does not depend on the particular choice of $\{t_k\}$ can be seen as follows: let $\{t'_k\}$ be another sequence going to ∞ and let μ'_ω be the corresponding limit. Considering the sequence $\{s_k\} = \{t_1, t'_1, t_2, t'_2, \dots\}$, we observe that the limit (1.5.4) with $t_k = s_k$ also exists almost surely such that $\mu_\omega = \mu'_\omega$ for almost all $\omega \in \Omega$.

We now show that

$$\mu(\mathrm{d}\omega, \mathrm{d}x) = \mu_\omega(\mathrm{d}x)\mathbb{P}(\mathrm{d}\omega)$$

is a Markov invariant measure for the RDS. By definition via (1.5.4), the disintegration $\{\mu_\omega\}$ is $\mathcal{F}^0_{-\infty}$-measurable. For proving invariance of μ, it is again enough to show that for any $t > 0$ and almost all $\omega \in \Omega$

$$\varphi(t, \omega, \cdot)^*\mu_\omega = \mu_{\theta_t\omega}.$$

Let us set $t_k = kt$ and choose any $\omega \in \tilde{\Omega}$ such that $\theta_t\omega \in \tilde{\Omega}$, which holds with probability one. Then we have by (1.5.4) and the cocycle property that for all $f \in C_b(X)$

$$\begin{aligned}
\langle f, \varphi(t, \omega, \cdot)^*\mu_\omega\rangle &= \int_X f(\varphi(t, \omega, x))\mu_\omega(\mathrm{d}x) = \lim_{k\to\infty} \int_X f\left(\varphi(t, \omega, \varphi(t_k, \theta_{-t_k}\omega, x))\right)\rho(\mathrm{d}x) \\
&= \lim_{k\to\infty} \int_X f\left(\varphi(t_{k+1}, \theta_{-t_{k+1}}(\theta_t\omega), x)\right)\rho(\mathrm{d}x) \\
&= \lim_{k\to\infty} \langle f, \varphi(t_k, \theta_{-t_k}(\theta_t\omega), \cdot)^*(\rho)\rangle = \langle f, \mu_{\theta_t\omega}\rangle.
\end{aligned}$$

It remains to show that $\mathbb{E}[\mu_\omega] = \rho$: take any $f \in C_b(X)$ and note that, due to (1.5.4), we have almost surely

$$\langle f, \mu_\omega\rangle = \lim_{k\to\infty} \langle f, \varphi(k, \theta_{-k}\omega, \cdot)^*\rho\rangle.$$

Taking the mean on bothe sides and using the dominated convergence theorem, we obtain

$$\begin{aligned}
\mathbb{E}\langle f, \mu_\omega\rangle &= \lim_{k\to\infty} \mathbb{E}\langle f, \varphi(k, \theta_{-k}\omega, \cdot)^*\rho\rangle \\
&= \lim_{k\to\infty} \mathbb{E}\langle f, \varphi(k, \omega, \cdot)^*\rho\rangle = \lim_{k\to\infty} \langle P_k f, \rho\rangle = \langle f, \rho\rangle,
\end{aligned}$$

where we have used the stationarity of ρ. This finishes the proof. □

Remark 1.5.7. We observe similarly to [5] that, in the situation of μ and ρ corresponding in the way described above,

$$\mathbb{E}[\mu_\omega(\cdot)|\mathcal{F}_0^\infty] = \mathbb{E}[\mu_\omega(\cdot)] = \rho(\cdot)\,,$$

and, hence,

$$\mathbb{E}[\mu(\cdot)|\mathcal{F}_0^\infty] = (\mathbb{P} \times \rho)(\cdot)\,.$$

Therefore the probability measure $\mathbb{P} \times \rho$ is invariant for $(\Theta_t)_{t\geq 0}$ on $(\Omega \times X, \mathcal{F}_0^\infty \times \mathcal{B}(X))$. In words, the product measure with marginals $\mathbb{P}$ and ρ is invariant for the random dynamical system restricted to one-sided path space; see also Question sheet 3. We will discuss a similar relation for quasi-stationary and quasi-ergodic measures in Chapter 5.

Chapter 2

(Linear and local) stability of random dynamical systems

In this chapter, we will prove Oseledets' Multiplicative Ergodic Theorem, which implies the existence of Lyapunov exponents along corresponding invariant subspaces, describing stability properties of a differentiable random dynamical system. Big parts of the proof and its preparation, in particular the proof of the subadditive ergodic theorem, are similar to [27].

2.1 Linear random dynamical systems and Lyapunov exponents

In the following, we will focus on the Euclidean state space $X = \mathbb{R}^d$. Assume that the random dynamical system (θ, φ) is C^k for some $k \geq 1$, i.e., the cocycle $\varphi(t, \omega, \cdot) \in C^k$ for all $t \in \mathbb{T}$ and $\omega \in \Omega$. The *linearization* or *derivative* $\mathrm{D}_x\varphi(t, \omega, x)$ of $\varphi(t, \omega, \cdot)$ at $x \in X$ is the Jacobian $d \times d$ matrix

$$\mathrm{D}_x\varphi(t, \omega, x) = \frac{\partial \varphi(t, \omega, x)}{\partial x} \,.$$

Differentiating the equation

$$\varphi(t + s, \omega, x) = \varphi(t, \theta_s\omega, \varphi(s, \omega, x))$$

on both sides and applying the chain rule to the right hand side yields

$$\mathrm{D}\varphi(t + s, \omega, x) = \mathrm{D}\varphi(t, \theta_s\omega, \varphi(s, \omega, x))\mathrm{D}\varphi(s, \omega, x) = \mathrm{D}\varphi(t, \Theta_s(\omega, x))\mathrm{D}\varphi(s, \omega, x) \,,$$

i.e. the cocycle property of $\mathrm{D}\varphi$ with respect to the skew product flow $(\Theta_t)_{t\in\mathbb{T}}$.

This will be our main example in mind when we work with linear random dynamical systems (Φ, θ) in the following.

Definition 2.1.1. A random dynamical system (θ, φ) is called *linear* if the map $\varphi(t, \omega) : \mathbb{R}^d \to \mathbb{R}^d$, $x \mapsto \varphi(t, \omega, x)$, is linear for any $(t, \omega) \in \mathbb{R} \times \Omega$. We then define $\Phi : \mathbb{R} \times \Omega \to \mathbb{R}^{d\times d}$ by $\Phi(t, \omega)x := \varphi(t, \omega, x)$.

In the following, we investigate the *Lyapunov exponents* $\lambda(\omega, x)$, defined, for all $0 \neq x \in \mathbb{R}^d$, by

$$\lambda(\omega, x) = \limsup_{t\to\infty} \frac{1}{t} \ln \|\Phi(t,\omega)x\|.$$

We will see that, under a suitable integrability assumption, these limits exist beyond the lim sup and that they correspond with invariant subspaces along the dynamics. A crucial theorem that we will need is discussed in the following section.

The main example throughout this lecture concerns stochastic differential equations in Stratonovich form, as given in (1.4.7), by

$$\mathrm{d}X_t = f_0(X_t)\mathrm{d}t + \sum_{j=1}^m g_j(X_t) \circ \mathrm{d}W_t^j$$

where W_t^j are independent real valued Brownian motions, f_0 is a C^1 vector field and $f_1, \dots, f_m$ are C^2 vector fields satisfying bounded growth conditions, as e.g. (global) Lipschitz continuity, in all derivatives to guarantee the existence of a (global) random dynamical system for φ and $\mathrm{D}\varphi$. We write the equation in Stratonovich form when differentiation is concerned as the classical rules of calculus are preserved. If the state space $X = \mathbb{R}^d$, we can apply the conversion formula to the Itô integral.

It is easy to observe analogously to Theorem 1.3.1, that the derivative $\mathrm{D}\varphi(t, \omega, x)$ applied to an initial condition $v_0 \in T_xX \cong \mathbb{R}^d$ solves uniquely the variational equation

$$\mathrm{d}v = \mathrm{D}f_0(\varphi(t,\omega,x))v\,\mathrm{d}t + \sum_{j=1}^m \mathrm{D}g_j(\varphi(t,\omega,x))v \circ \mathrm{d}W_t^j\,, \quad v(0) = v_0 \in T_xX\,. \tag{2.1.1}$$

In case the derivative can be written as a matrix, as for example for $X = \mathbb{R}^d$, the Jacobian $\mathrm{D}\varphi(t, \omega, x)$ satisfies Liouville's equation

$$\begin{aligned} \det \mathrm{D}\varphi(t,\omega,x) = \exp\Bigg(&\int_0^t \operatorname{trace} \mathrm{D}f_0(\varphi(s,\omega,x))\mathrm{d}s \\ &+ \sum_{j=1}^m \int_0^t \operatorname{trace} \mathrm{D}f_j(\varphi(s,\omega,x)) \circ \mathrm{d}W_s^j \Bigg). \end{aligned} \tag{2.1.2}$$

2.2 Kingman's subadditive ergodic theorem

In this section, we work in discrete time, and consider $(\Omega, \mathcal{F}, \mathbb{P})$ with measure-preserving $\theta : \Omega \to \Omega$. For a measurable function $\phi : \Omega \to [-\infty, \infty]$ we will write

$$\phi^+(\omega) = \max\{0, \phi(\omega)\}, \quad \phi^-(\omega) = \max\{0, -\phi(\omega)\}.$$

A measurable function ϕ is called *invariant* if $\phi(\theta\omega) = \phi(\omega)$ for $\mathbb{P}$-almost all $\omega \in \Omega$.

Definition 2.2.1. A sequence $\phi_n : \Omega \to [-\infty, \infty], n \geq 1$ of measurable functions is *subadditive*, relative to θ, if

$$\phi_{n+m} \leq \phi_m + \phi_n \circ \theta^m \quad \text{for all } m, n \geq 1.$$

Example 2.2.2. *1. For any measurable function $\psi : \Omega \to \mathbb{R}$, the orbital sum $\phi_n = \sum_{j=0}^{n-1} \psi \circ \theta^j$ is subadditive (even additive).*

2. Given any measurable map $A : \Omega \to \mathbb{R}^{d\times d}$, consider the sequence $\phi_n(\omega) = \log \|A^n(\omega)\|$, where

$$A^n(\omega) = A(\theta^{n-1}\omega) \cdots A(\theta\omega)A(\omega).$$

By sublinearity of matrix products, you may check that the sequence is subadditive. This is precisely the example that will be important for the Furstenberg-Kesten and the Multiplicative Ergodic Theorem.

We can now formulate the main theorem of this section:

Theorem 2.2.3 (Kingman's subadditive ergodic theorem)**.** *Let $\phi_n : \Omega \to [-\infty, \infty)$, $n \geq 1$, be a subadditive sequence of measurable functions such that $\phi_1^+ \in L^1(\mathbb{P})$. Then (ϕ_n/n) converges $\mathbb{P}$-almost everywhere to some invariant function $\phi : \Omega \to [-\infty, \infty)$ (which is constant when $\mathbb{P}$ is ergodic).*

Furthermore, the positive part ϕ^+ is integrable and

$$\int \phi \, \mathrm{d}\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int \phi_n \, \mathrm{d}\mathbb{P} = \inf_n \frac{1}{n} \int \phi_n \, \mathrm{d}\mathbb{P} \in [-\infty, +\infty).$$

[**End of Lecture IV, 04.05.**]

Proof. Firstly, observe that, by subadditivity and induction,

$$\phi_n \leq \phi_1 + \phi_1 \circ \theta + \cdots + \phi_1 \circ \theta^{n-1}.$$

The same holds true if we replace ϕ_n and ϕ_1 by ϕ_n^+ and ϕ_1^+. Hence, $\phi_1^+ \in L^1(\mathbb{P})$ implies that $\phi_n^+ \in L^1(\mathbb{P})$ for all n.

Moreover, observe that

$$a_n = \int \phi_n \, \mathrm{d}\mathbb{P}$$

is a subadditive sequence in $[-\infty, \infty)$. Hence, by Exercise 1 on Question Sheet 4, we have that

$$\lim_{n\to\infty} \frac{a_n}{n} = \inf_{n\in\mathbb{N}} \frac{a_n}{n} =: L \in [-\infty, \infty)$$

exists (see also QS 4, Exercise 3 (a)). Define now $\phi_- : \Omega \to [-\infty, \infty]$ and $\phi_+ : \Omega \to [-\infty, \infty]$ by

$$\phi_-(\omega) = \liminf_{n\to\infty} \frac{\phi_n}{n}(\omega), \quad \phi_+(\omega) = \limsup_{n\to\infty} \frac{\phi_n}{n}(\omega).$$

We will show that

$$\int \phi_- \, \mathrm{d}\mathbb{P} \geq L \geq \int \phi_+ \, \mathrm{d}\mathbb{P}, \tag{2.2.1}$$

provided that every ϕ_m is bounded away from $-\infty$. Hence, we obtain the statement of the theorem for this case with $\phi = \phi_- = \phi_+$, which is invariant (see QS 4, Exercise 3 (c)). At the end, we use a truncation trick to remove the boundedness assumption and obtain the proof in full generality.

1. We firstly make an important estimate for ϕ_n: Assume in the following that $\phi_- > -\infty$ at every point, and fix $\varepsilon > 0$. We define, for each $k \in \mathbb{N}$,

$$E_k = \{\omega \in \Omega \,:\, \phi_j(\omega) \leq j(\phi_-(\omega) + \varepsilon) \text{ for some } j \in \{1, \dots, k\}\}.$$

Note that $E_k \subset E_{k+1}$ for all k and $\Omega = \cup_k E_k$. We define

$$\psi_k(\omega) = \begin{cases} \phi_-(\omega) + \varepsilon, & \text{if } \omega \in E_k, \\ \phi_1(\omega), & \text{if } \omega \in E_k^c. \end{cases}$$

Observing that $\psi_k(\omega)$ decreases to $\phi_-(\omega) + \varepsilon$ as $k \to \infty$, for all $\omega \in \Omega$, we obtain by the monotonic convergence theorem

$$\int \psi_k \,\mathrm{d}\mathbb{P} \to \int (\phi_- + \varepsilon) \,\mathrm{d}\mathbb{P}, \quad \text{as } k \to \infty.$$

The estimate is

$$\phi_n(\omega) \leq \sum_{i=0}^{n-k+1} \psi_k(f^i(\omega)) + \sum_{i=n-k}^{n-1} \max\{\psi_k, \phi_1\}(\theta^i \omega), \tag{2.2.2}$$

for any $n > k \geq 1$ and $\mathbb{P}$-almost all $\omega \in \Omega$.

Proof of estimate (2.2.2): For almost all $\omega \in \Omega$, we have $\phi_-(\omega) = \phi_-(\theta^j \omega))$ for any $j \geq 1$ (QS 4, Exercise 3 (c)). Taking such ω, consider now the, possibly finite, sequence

$$m_0 \leq n_1 < m_1 \leq n_2 < m_2 < \cdots$$

defined in the following way: Take $m_0 = 0$. Given $j \geq 1$, let n_j be the smallest integer greater or equal to m_{j-1} that satisfies $\theta^{n_j} \omega \in E_k$ (assuming it exists). Then there is an m_j with $1 \leq m_j - n_j \leq k$ such that

$$\phi_{m_j - n_j} (\theta^{n_j} \omega) \leq (m_j - n_j) \left[\phi_- (\theta^{n_j} \omega) + \varepsilon\right]. \tag{2.2.3}$$

Now given any $n \geq k$, let $l \geq 0$ be the largest integer such that $n \geq m_l$. Subadditivity yields

$$\phi_{n_j - m_{j-1}} (\theta^{m_{j-1}} \omega) \leq \sum_{i=m_{j-1}}^{n_j - 1} \phi_1(\theta^i \omega)$$

for any $j = 1, \dots, l$ such that $m_{j-1} \neq n_j$, and similarly for $\phi_{n-m_l}(\theta^{m_l}\omega)$. We observe that

$$\begin{aligned}\phi_n(\omega) &\leq \phi_{m_l}(\omega) + \phi_{n-m_l}(\theta^{m_l}\omega) \\ &\leq \sum_{j=1, n_j \neq m_{j-1}}^{l} \phi_{n_j - m_{j-1}}(\theta^{m_{j-1}}\omega) + \sum_{j=1}^{l} \phi_{m_j - n_j}(\theta^{n_j}\omega) + \phi_{n-m_l}(\theta^{m_l}\omega) \\ &\leq \sum_{i \in I} \phi_1(\theta^i\omega) + \sum_{j=1}^{l} \phi_{m_j - n_j}(\theta^{n_j}\omega), \end{aligned} \tag{2.2.4}$$

where $I = \left(\cup_{j=1}^{l}[m_{j-1}, n_j) \cup [m_l, n)\right) \cap \mathbb{N}$. Observe that

$$\phi_1(\theta^i\omega) = \psi_k(\theta^i\omega)$$

for any

$$i \in I = \left(\cup_{j=1}^{l}[m_{j-1}, n_j) \cup [m_l, \min\{n_{l+1}, n\})\right) \cap \mathbb{N},$$

since $\theta^i\omega \in E_k^c$ in these cases. At the same time, using the invariance of ϕ_- and the fact that $\psi_k \geq \phi_- + \varepsilon$, we observe from (2.2.3) that

$$\phi_{m_j - n_j}\left(\theta^{n_j}\omega\right) \leq \sum_{i=n_j}^{m_j - 1} \left[\phi_-\left(\theta^i\omega\right) + \varepsilon\right] \leq \sum_{i=n_j}^{m_j-1} \psi_k\left(\theta^i\omega\right)$$

for every $j = 1, \dots, l$. We can combine this with (2.2.4) to obtain

$$\phi_n \leq \sum_{i=0}^{\min\{n_{l+1}, n\}-1} \psi_k\left(\theta^i\omega\right) + \sum_{i=n_{l+1}}^{n-1} \phi_1\left(\theta^i\omega\right)$$

Since $n_{l+1} \geq m_{l+1} - k > n - k$, estimate (2.2.2) is shown.

2. We now use the estimate (2.2.2) to show

$$\int \phi_- \, \mathrm{d}\mathbb{P} = L, \tag{2.2.5}$$

whenever ϕ_n/n is uniformly bounded from below, i.e., there exists $\kappa > 0$ such that $\phi_n/n \geq -\kappa$ for every n, such that, in particular, $\phi_{\geq} - \kappa > -\infty$.

In this situation, we can apply Fatou's lemma to $\phi_n/n + \kappa$ to obtain that ϕ_- is integrable with

$$\int \phi_- \, \mathrm{d}\mathbb{P} \leq \lim_{n\to\infty} \int \frac{\phi_n}{n} \, \mathrm{d}\mathbb{P} = L.$$

For the opposite inequality we observe from (2.2.2) that

$$\frac{1}{n}\int \phi_n \, \mathrm{d}\mathbb{P} \leq \frac{n-k}{n} \int \psi_k \mathrm{d}\mathbb{P} + \frac{k}{n} \int \max\{\psi_k, \phi_1\} \, \mathrm{d}\mathbb{P}.$$

Noting that $\max\{\psi_k, \phi_1\} \leq \max\{\phi_- + \varepsilon, \phi_1^+\}$ and taking $n \to \infty$, we obtain that $L \leq \int \psi_k \, \mathrm{d}\mathbb{P}$

for any k. Now, letting $k \to \infty$, we get

$$L \le \int \phi_- \, \mathrm{d}\mathbb{P} + \varepsilon.$$

Since ε could be taken arbitrarily small, the proof is finished for this case.

Now, removing the uniform boundedness assumption, we define for each $\kappa > 0$,

$$\phi_n^\kappa = \max\{\phi_n, -\kappa n\}, \quad \text{and} \quad \phi_-^\kappa = \max\{\phi_-, -\kappa\}.$$

The sequence ϕ_n^κ is clearly subadditive, the positive part of ϕ_1^κ is integrable, we clearly have $\phi_-^\kappa = \liminf_{n\to\infty}(1/n)\phi_n^\kappa$ and the uniform boundedness assumption holds. Hence, for any fixed $\kappa > 0$, we deduce from the already shown that

$$\int \phi_-^\kappa \, \mathrm{d}\mathbb{P} = \inf_n \frac{1}{n} \int \phi_n^\kappa \, \mathrm{d}\mathbb{P}. \tag{2.2.6}$$

By monotone convergence, we additionally observe that

$$\int \phi_n \, \mathrm{d}\mathbb{P} = \inf_\kappa \int \phi_n^\kappa \, \mathrm{d}\mathbb{P} \quad \text{and} \quad \int \phi_- \, \mathrm{d}\mathbb{P} = \inf_\kappa \int \phi_-^\kappa \, \mathrm{d}\mathbb{P}. \tag{2.2.7}$$

Now, combining (2.2.6) and (2.2.7), we obtain

$$\int \phi_- \, \mathrm{d}\mathbb{P} = \inf_\kappa \int \phi_-^\kappa \, \mathrm{d}\mathbb{P} = \inf_\kappa \inf_n \frac{1}{n} \int \phi_n^\kappa \, \mathrm{d}\mathbb{P} = \inf_n \frac{1}{n} \int \phi_n \, \mathrm{d}\mathbb{P} = L,$$

which shows the claim (2.2.5).

3. We have shown on Question Sheet 4, Exercise 3 (d), that, if $\inf_{\omega\in\Omega} \phi_n(\omega) > -\infty$ for any n, we may now deduce that

$$\int \phi_+ \, \mathbb{P} \le L.$$

This shows all the claims of the theorem under the boundedness assumptions.

4. Finally, for the general case, define again for any $\kappa > 0$

$$\phi_n^\kappa = \max\{\phi_n, -\kappa n\}, \quad \text{and} \quad \phi_-^\kappa = \max\{\phi_-, -\kappa\}, \quad \text{and} \quad \phi_+^\kappa = \max\{\phi_+, -\kappa\}.$$

Then the previous arguments hold clearly for the sequence ϕ_n^κ for any fixed $\kappa > 0$ such that $\phi_+^\kappa = \phi_-^\kappa$ almost surely. At the same time, we have that

$$\phi_-^\kappa \to \phi_- \quad \text{and} \quad \phi_+^\kappa \to \phi_+ \quad \text{when} \quad \kappa \to \infty,$$

such that $\phi_- = \phi_+$ almost surely. This finishes the proof of the theorem, where the fact that ϕ is constant almost surely when $\mathbb{P}$ is ergodic, follows from the classical fact invariant functions are constant almost surely with respect to ergodic measures. □

We now obtain the famous ergodic theorem by Birkhoff as a corollary:

Corollary 2.2.4 (Birkhoff's ergodic theorem)**.** *Let $\phi : \Omega \to \mathbb{R}$ be a $\mathbb{P}$-integrable function. Then for almost all ω there is*

$$\tilde{\phi}(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \phi(\theta^j \omega),$$

such that $\tilde{\phi}$ is invariant and integrable with

$$\int \tilde{\phi} \, \mathrm{d}\mathbb{P} = \int \phi \, \mathrm{d}\mathbb{P}.$$

If $\mathbb{P}$ is ergodic, we have for almost all ω

$$\tilde{\phi} = \int \phi \, \mathrm{d}\mathbb{P}.$$

Proof. The statements follow directly from Theorem 2.2.3, considering the (sub-)additive sequence $\sum_{j=0}^{n-1} \phi(\theta^j \omega)$. ☐

2.3 Theorem about extremal Lyapunov exponents

Consider in the following the setting of a linear random dynamical system in discrete time with skew product $\Theta : \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d$ given by $\Theta(\omega, v) = (\theta\omega, A(\omega)v)$, where $A : \Omega \to \mathbb{R}^{d\times d}$ is a (invertible) matrix generating the RDS. In particular, we will write the cocycle as

$$\Phi(n, \omega) = A^n(\omega) = A(\theta^{n-1}\omega) \cdots A(\theta\omega)A(\omega).$$

In the following we will write for any $x \in (0, \infty)$

$$\log^+(x) = \max\{0, \log(x)\}.$$

From the previous subsection it is now easy to infer the following theorem about extremal Lyapunov exponents, in some cases also called the Furstenberg-Kesten theorem:

Theorem 2.3.1 (Furstenberg-Kesten)**.** *If $\log^+ \|A^{\pm 1}\| \in L^1(\mathbb{P})$, then the extremal Lyapunov exponents*

$$\lambda_+(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)\| \quad \textit{and} \quad \lambda_-(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|(A^n(\omega))^{-1}\|^{-1}$$

exist for almost all $\omega \in \Omega$.

Furthermore, the functions $\lambda_\pm$ are invariant and $\mathbb{P}$-integrable, with

$$\int \lambda_+ \, \mathrm{d}\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int \log \|A^n(\omega)\| \, \mathrm{d}\mathbb{P},$$
$$\int \lambda_- \, \mathrm{d}\mathbb{P} = \lim_{n\to\infty} \frac{1}{n} \int \log \|(A^n(\omega))^{-1}\|^{-1} \, \mathrm{d}\mathbb{P}.$$

Proof. Define

$$\phi_n = \log \|A^n(\omega)\|$$

and

$$\psi_n = \log \|(A^n(\omega))^{-1}\|.$$

Then, by the assumptions, we have that $\phi_1^+, \psi_1^+ \in L^1(\mathbb{P})$ and, hence, $\phi_1(\omega), \psi_1(\omega) \in [-\infty, \infty)$ for almost all $\omega \in \Omega$. It is easy to see by sub-multiplicity of the matrix norm and the properties of log that the sequences ϕ_n, ψ_n are subadditive. Hence, the statements follow from Subadditive Ergodic Theorem 2.2.3. □

[**End of Lecture V, 11.05.**]

2.4 The Multiplicative Ergodic Theorem in two dimensions

Firstly, let us state and prove the theorem in two dimensions as it possible to already see the main features without high technical effort.

Consider in the following the setting of a linear random dynamical system in discrete time and two space dimensions, with skew product $\Theta : \Omega \times \mathbb{R}^2 \to \Omega \times \mathbb{R}^2$ given by $\Theta(\omega, v) = (\theta\omega, A(\omega)v)$, where $A : \Omega \to \mathbb{R}^{2\times 2}$ is an invertible matrix generating the RDS. In particular, we will write the cocycle as

$$\Phi(n, \omega) = A^n(\omega) = A(\theta^{n-1}\omega) \cdots A(\theta\omega)A(\omega).$$

Recall the definition of the extremal Lyapunov exponents λ_+, λ_- from above.

Theorem 2.4.1 (One-sided MET in two dimensions)**.** *Consider a measurable family of invertible matrices $A : \Omega \to \mathbb{R}^{2\times 2}$ such that*

$$\log^+ \|A^{\pm 1}\| \in L^1(\mathbb{P})$$

is satisfied. Then for almost all $\omega \in \Omega$ the Lyapunov exponents λ_-, λ_+ satisfy

(1) either $\lambda_-(\omega) = \lambda_+(\omega)$ and

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \lambda_\pm(\omega), \quad \textit{for all } v \in \mathbb{R}^2.$$

(2) or $\lambda_+(\omega) > \lambda_-(\omega)$ and there exists a vector line $E^s(\omega) \subset \mathbb{R}^2$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \begin{cases} \lambda_-(\omega), & \textit{if } 0 \neq v \in E^s(\omega), \\ \lambda_+(\omega), & \textit{if } 0 \neq v \in \mathbb{R}^2 \setminus E^s(\omega). \end{cases}$$

Furthermore, $A(\omega)E^s(\omega) = E^s(\theta\omega)$.

Proof. For the proof we refer to Question Sheet 5. □

In two-sided time, we even obtain a full decomposition of $\mathbb{R}^2$ with respect to the dynamics.

Theorem 2.4.2 (Two-sided MET in two dimensions)**.** *Assume the situation of Theorem 2.4.1 with the additional assumption that $\theta : \Omega \to \Omega$ is invertible almost surely. Then for almost all $\omega \in \Omega$ the Lyapunov exponents λ_-, λ_+ satisfy*

(1) either $\lambda_-(\omega) = \lambda_+(\omega)$ and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \lambda_\pm(\omega), \quad \text{for all } v \in \mathbb{R}^2.$$

(2) or $\lambda_+(\omega) > \lambda_-(\omega)$ and there exists a direct sum decomposition $\mathbb{R}^2 = E^s(\omega) \oplus E^u(\omega)$ such that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \begin{cases} \lambda_-(\omega), & \text{if } 0 \neq v \in E^s(\omega), \\ \lambda_+(\omega), & \text{if } 0 \neq v \in \mathbb{R}^2 \setminus E^s(\omega), \end{cases}$$

and

$$\lim_{n\to-\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \begin{cases} \lambda_-(\omega), & \text{if } 0 \neq v \in \mathbb{R}^2 \setminus E^u(\omega), \\ \lambda_+(\omega), & \text{if } 0 \neq v \in E^u(\omega). \end{cases}$$

Furthermore, $A(\omega)E^s(\omega) = E^s(\theta\omega)$ and $A(\omega)E^u(\omega) = E^u(\theta\omega)$, and

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\sin \measuredangle(E^u(\theta^n\omega), E^s(\theta^n\omega))| = 0.$$

Proof. Similarly to QS 5, we deal only with $A(\omega) \in \mathrm{SL}(2)$ since the statement can then easily be transferred to the general setting. Let us write again $\lambda(\omega) = \lambda_+(\omega) = -\lambda_-(\omega)$.

The case $\lambda(\omega) = 0$ directly follows from Theorem 2.4.1 applied to the RDS Θ and its inverse Θ^{-1}. Hence, from now on we assume that $\lambda(\omega) > 0$. From Theorem 2.4.1, we can take $E^s(\omega) = \mathbb{R}s(\omega)$ and $E^u(\omega) = \mathbb{R}u(\omega)$ for Θ and Θ^{-1} respectively. Hence, we need to show that the vectors $s(\omega)$ and $u(\omega)$ are non-collinear for almost all ω such that $\lambda(\omega) > 0$.

For that, it is enough to show that

$$\lim_{n\to-\infty} \frac{1}{n} \log \|A^n(\omega)|E^s(\omega)\| = -\lambda(\omega),$$

since we have $\lim_{n\to-\infty} \frac{1}{n} \log \|A^n(\omega)|E^u(\omega)\| = \lambda(\omega)$ from applying Theorem 2.4.1 to Θ^{-1}. Let us denote

$$\psi(\omega) := \lim_{n\to-\infty} \frac{1}{n} \log \|A^n(\omega)|E^s(\omega)\|,$$

which exists by applying Theorem 2.4.1 to Θ^{-1}, and consider the sequences of functions

$$\psi_n(\omega) := \frac{1}{-n} \log \|A^{-n}(\omega)|E^s(\omega)\| \quad \text{and} \quad \phi_n(\omega) := \frac{1}{-n} \log \|(A^n(\omega)|E^s(\omega))^{-1}\|.$$

Note from the definition of A^{-n} that $\psi_n(\omega) = \phi_n(\theta^{-n}\omega)$ for all $n \geq 1$. Since E^s is one-dimensional, we may even write $\phi_n(\omega) = \frac{1}{n} \log \|A^n(\omega)|E^s(\omega)\|$, such that we can see that

$\lim_{n\to\infty} \phi_n(\omega) = -\lambda(\omega)$. In particular, we have that

$$\lim_{n\to\infty} \mathbb{P}\{\omega : |\phi_n(\omega) + \lambda(\omega)| > \delta\} = 0 \text{ for any } \delta > 0.$$

Using the measure invariance of θ, we conclude

$$\lim_{n\to\infty} \mathbb{P}\{\omega : \left|\phi_n(\theta^{-n}\omega) + \lambda(\theta^{-n}\omega)\right| > \delta\} = 0 \text{ for any } \delta > 0.$$

With the above and the invariance of λ we obtain

$$\lim_{n\to\infty} \mathbb{P}\{\omega : |\psi_n(\omega) + \lambda(\omega)| > \delta\} = 0 \text{ for any } \delta > 0,$$

i.e. ψ_n converges to $-\lambda$ in probability. Using that ψ_n converges to ψ almost everywhere, we obtain $\psi = \lambda_-$ as claimed.

It remains to show that, for $\alpha(\omega) := \measuredangle(E^s(\omega), E^u(\omega))$, we get

$$\lim_{n\to\pm\infty} \frac{1}{n} \log|\sin\alpha(\theta^n\omega)| = 0.$$

It is an elementary exercise to show that

$$\|A(\omega)\|^{-2} \le \frac{|\sin\alpha(\theta\omega)|}{|\sin\alpha(\omega)|} \le \|A(\omega)\|^2.$$

Hence, we observe that

$$|\log|\sin\alpha(\theta\omega)| - \log|\sin\alpha(\omega)|| \le 2\log\|A(\omega)\|,$$

such that $\log|\sin\alpha| \circ \theta - \log|\sin\alpha| \in L^1(\mathbb{P})$. In combination with Exercise 1, QS 5, this finishes the proof. □

2.5 The MET in arbitrary dimensions

For the following, let $\Theta_t : \Omega \times \mathbb{R}^d \to \Omega \times \mathbb{R}^d$ be a linear RDS given as skew product $\Theta_t(\omega, x) = (\theta_t\omega, \Phi(t,\omega)x)$, where t can be discrete or continuous time. In discrete time, we will have as before a generating matrix $A(\omega) = \Phi(1,\omega)$. In contiuous time, you can think of $\Phi(t,\omega) = \mathrm{D}_y\varphi(t,\tilde\omega,y)$ where $\varphi(t,\tilde\omega,y)$ solves a random or stochastic differential equation and Φ is a cocycle over

$$\tilde\Theta_t : \Omega := \tilde\Omega \times \mathbb{R}^d \to \tilde\Omega \times \mathbb{R}^d, \quad \tilde\Theta_t(\omega) = \Theta_t(\tilde\omega, y) = (\tilde\theta_t\tilde\omega, \varphi(t,\tilde\omega,y),$$

which itself is an RDS with invariant measure $\tilde{\mathbb{P}}$.

For the following we need the following notions: a Grassmannian manifold $\mathrm{Gr}(l,d)$, $0 \le l \le d$, is the set of all l-dimensional linear subspaces of $\mathbb{R}^d$. The Grassmannian of $\mathbb{R}^d$ is the disjoint union $\mathrm{Gr}(d)$ of the Grassmanian manifolds $\mathrm{Gr}(l,d)$. A map $\omega \mapsto V_\omega$ with values in $\mathrm{Gr}(d)$ is measurable if and only if there exist measurable, linearly independent vector fields that span V_ω

at each point (see Question Sheet 6). Furthermore, we have the following definition:

Definition 2.5.1. A *flag* in $\mathbb{R}^d$ is a decreasing family $W^1 \supsetneq \cdots \supsetneq W^k \supset \{0\}$ of vector subspaces of $\mathbb{R}^d$. The flag is called complete if $k = d$ and $\dim W^j = d + 1 - j$ for all $j = 1, \ldots, d$.

We can now use all our observations from before to prove the following major theorems of the course. We will firstly state both theorems, the Multiplicative Ergodic Theorem in one-sided and in two-sided time, and then give (sketches) of the proof afterwards.

Theorem 2.5.2 (Oseledets MET in one-sided time)**.** *Consider a linear RDS $\Theta = (\theta, \Phi)$ on $\Omega \times \mathbb{R}^d$, i.e. $\Phi(t,\omega) : \mathbb{R}^d \to \mathbb{R}^d$ is a linear cocycle over the metric DS $(\Omega, \mathcal{F}, \mathbb{P}, \theta_t)$. Then the following statements holds:*

(A) Let time $\mathbb{T} = \mathbb{N}$ and the generator satisfy

$$\log^+ \|A^{\pm 1}\| \in L^1(\mathbb{P}). \tag{2.5.1}$$

Then for almost every ω there are $k = k(\omega)$, numbers $\lambda_1(\omega) > \cdots > \lambda_k(\omega)$ and a flag $\mathbb{R}^d = V_\omega^1 \supsetneq \cdots \supsetneq V_\omega^k \supsetneq \{0\}$ such that for all $i = 1, \ldots, k$:

(a) $k(\theta\omega) = k(\omega)$, $\lambda_i(\theta\omega) = \lambda_i(\omega)$ and $A(\omega)V_\omega^i = V_{\theta\omega}^i$;

(b) the maps $\omega \mapsto k(\omega)$, $\omega \mapsto \lambda_i(\omega)$ and $\omega \mapsto V_\omega^i$ with values in $\mathbb{N}$, $\mathbb{R}$ and $\mathrm{Gr}(d)$, respectively, are measurable;

(c)

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \lambda_i(\omega) \quad \text{for all } v \in V_\omega^i \setminus V_\omega^{i+1} \qquad (\text{with } V_\omega^{k+1} = \{0\}).$$

If $\mathbb{P}$ is ergodic, then $k(\omega)$ and each Lyapunov exponent *$\lambda_i(\omega)$ are constant almost surely, and so are the dimensions of the* Oseledets subspaces *V_ω^i.*

(B) Let $\mathbb{T} = \mathbb{R}^+$ and $\Phi(t,\omega) \in \mathrm{GL}(d)$. If we have

$$\sup_{0 \le t \le 1} \log^+ \|\Phi(t,\cdot)^{\pm 1}\| \in L^1(\mathbb{P}), \tag{2.5.2}$$

then all statements of part (A) hold with n, θ and $A^n(\omega)$ replaced with t, θ_t and $\Phi(t,\omega)$.

We call $\dim V_\omega^i - \dim V_\omega^{i+1}$ the *multiplicity* of the corresponding Lyapunov exponent. The *Lyapunov spectrum* of the RDS is the set of all Lyapunov exponents, each counted with multiplicity. The Lyapunov exponent is simple if all Lyapunov exponents have multiplicity 1 or, equivalently, if the Oseledets flag is complete.

In two-sided time, we get the even stronger statement:

Theorem 2.5.3 (Oseledets MET in two-sided time)**.** *Consider the situation of Theorem 2.5.2 with now $\theta_t : \Omega \to \Omega$ being invertible. Then the following statements hold:*

(A) Let time $\mathbb{T} = \mathbb{N}$ and the generator satisfy (2.5.1)*. Then for almost every ω there exists a direct sum decomposition $\mathbb{R}^d = E^1(\omega) \oplus \cdots \oplus E^k(\omega)$ such that, for all $i = 1, \ldots, k$:*

(a)

$$A(\omega)E^i(\omega) = E^i(\theta\omega) \quad \textit{and} \quad V_\omega^i = \bigoplus_{j=i}^{k} E^j(\omega),$$

(b)

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(\omega)v\| = \lambda_i(\omega) \quad \textit{for all } v \in E^i(\omega) \setminus \{0\},$$

(c)

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left| \sin \measuredangle \left(\bigoplus_{i\in I} E^i(\theta^n\omega), \bigoplus_{j\in J} E^j(\theta^n\omega) \right) \right| = 0 \quad \textit{when } I \cap J = \emptyset,$$

where the angle between two subspaces is the smallest angle between non-zero vectors of these subspaces.

(B) Let $\mathbb{T} = \mathbb{R}^+$ and $\Phi(t,\omega) \in \mathrm{GL}(d)$. If we have

$$\sup_{0\le t\le 1} \log^+ \|\Phi(t,\cdot)^{\pm 1}\| \in L^1(\mathbb{P}), \tag{2.5.3}$$

then all statements of part (A) hold with n, θ and $A^n(\omega)$ replaced with t, θ_t and $\Phi(t,\omega)$.

Note that the multiplicity of each Laypunov exponent λ_i coincides with the dimension $\dim E^i(\omega) = \dim V_\omega^i - \dim V_\omega^{i+1}$ of the associated Oseledets space $E^i(\omega)$. Thus the Lyapunov spectrum is simple if and only if $\dim E^i(\omega) = 1$ for all i.

Proof of Theorem 2.5.2. (A) We will focus on the proof for discrete time and sketch how one extends to continuous time later.

1. Constructing the Oseledets flag: For each $v \in \mathbb{R}^d \setminus \{0\}$ and $\omega \in \Omega$, we set

$$\lambda(\omega, v) = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\|. \tag{2.5.4}$$

Recall the extremal Lyapunov exponents $\lambda_\pm(\omega)$ and observe the following properties:

Lemma 2.5.4. *For almost all $\omega \in \Omega$ and any $v, v' \in \mathbb{R}^d \setminus \{0\}$,*

(i) $\lambda_-(\omega) \le \lambda(\omega, v) \le \lambda_+(\omega)$;

(ii) $\lambda(\omega, cv) = \lambda(\omega, v)$ *for* $c \neq 0$;

(iii) $\lambda(\omega, v + v') = \max\{\lambda(\omega, v), \lambda(\omega, v')\}$ *if* $v + v' \neq 0$;

(iv) $\lambda(\omega, v) = \lambda(\theta\omega, A(\omega)v)$.

This lemma is shown in Exercise 2, QS 6.

Let us now take ω as in the above Lemma and $k(\omega) \ge 1$ to be the number of elements of the set

$$L := \{\lambda(\omega, v) \,:\, v \in \mathbb{R}^d \setminus \{0\}\}.$$

Let $\lambda_1(\omega) > \cdots > \lambda_{k(\omega)}$ be those elements and define

$$V_\omega^i = \{v \in \mathbb{R}^d \setminus \{0\} \,:\, \lambda(\omega, v) \le \lambda_i(\omega)\} \cup \{0\} \quad \text{for } i = 1, \dots, k(\omega).$$

From Lemma 2.5.4, we obtain that V_ω^i is a vector space for every i. Furthermore, we get the flag property from these definitions, i.e.,

$$\mathbb{R}^d = V_\omega^1 \supsetneq \cdots \supsetneq V_\omega^{k(\omega)} \supsetneq \{0\},$$

where $k(\omega) \le d$. We also observe that

$$\lambda(\omega, v) = \lambda_i(\omega) \quad \text{for every } v \in V_\omega^i \setminus V_\omega^{i+1},$$

and

$$\lambda_-(\omega) \le \lambda_i(\omega) \le \lambda_+(\omega) \quad \text{for all } i = 1, \dots, k(\omega).$$

Finally, we obtain the invariance of $k(\omega), \lambda_i(\omega)$ and V_ω^i with respect to (θ, A) from Lemma 2.5.4 (iv), and the fact that $A(\omega)$ is a bijection. Note in particular that we need the invertibility of $A(\omega)$ at precisely this point.

We have shown part (a) of the theorem.

2. Measurability: For the proof of measurability, as stated in (b), we need the following statements (for proofs see Castaing and Valadier [7]):

Proposition 2.5.5 (Theorem III.23 in [7])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and Y be a separable metric space. Let further $\mathcal{F} \otimes \mathcal{B}(Y)$ be the product σ-algebra in $\Omega \times Y$ and $\pi : \Omega \times Y$ be the canonical projection. Then $\pi(E) \in \mathcal{F}$ for every $E \in \mathcal{F} \otimes \mathcal{B}(Y)$.*

Proposition 2.5.6 (Theorem III.30 in [7])**.** *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and Y be a separable metric space. Let $\mathcal{K}(Y)$ be the space of compact subsets of Y, with the Haudorff topology. The following are equivalent:*

(a) a map $\omega \mapsto K_\omega$ from Ω to $\mathcal{K}(Y)$ is measurable;

(b) the graph $\{(\omega, y) \,:\, y \in K_\omega\}$ is in $\mathcal{F} \otimes \mathcal{B}(Y)$;

(c) $\{\omega \in \Omega \,:\, K_\omega \cap U \neq \emptyset\} \in \mathcal{F}$ for any open set $U \subset Y$.

We obtain the following as a corollary of Proposition 2.5.6:

Proposition 2.5.7. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space and $\omega \mapsto V_\omega$ be a map from Ω to the Grassmannian $\mathrm{Gr}(d)$. Then the following are equivalent:*

(a) the map $\omega \mapsto V_\omega$ is measurable

(b) the graph $\{(\omega, v) \in \Omega \times \mathbb{R}^d \,:\, v \in V_\omega\}$ is in $\mathcal{F} \times \mathcal{B}(\mathbb{R}^d)$.

Without loss of generality, we may assume that our probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is complete (otherwise it can be completed). Hence, we can apply these propositions.

Let $e_1, \dots e_d$ be an arbitrary basis of $\mathbb{R}^d$ and note with Lemma 2.5.4 that

$$\lambda_1(\omega) = \max\{\lambda(\omega, e_i) \,:\, 1 \leq i \leq d\}.$$

Since $(\omega, v) \mapsto \lambda(\omega, v)$ is measurable, it follows that $\omega \mapsto \lambda_1(\omega)$ is measurable and

$$V_*^2 = \{(\omega, v) \in \Omega \times \mathbb{R}^d \setminus \{0\} : \lambda(\omega, v) < \lambda_1(\omega)\}$$

is a measurable subset of $\Omega \times \mathbb{R}^d$. Observe that

$$\begin{aligned}\pi(V_*^2) &= \{\omega \in \Omega \,:\, \lambda(\omega, v) < \lambda_1(\omega) \text{ for some } v \in \mathbb{R}^d \setminus \{0\}\} \\ &= \{\omega \in \Omega \,:\, k(\omega) \geq 2\}.\end{aligned}$$

By Proposition 2.5.5 this is a measurable subset of Ω. For $\omega \in \pi(V_*^2)$ we define

$$V_\omega^2 = \{v \in \mathbb{R}^d \,:\, (\omega, v) \in V_*^2\} \cup \{0\}.$$

Since $V_*^2 \cup (\Omega \times \{0\})$ is a measurable subset of $\Omega \times \mathbb{R}^d$, Proposition 2.5.7 gives that $\omega \to V_\omega^2$ is a measurable map on $\pi(V_*^2)$. Hence, by Exercise 1 on Question Sheet 6, each

$$\Omega_l^2 = \{\omega \in \pi(V_*^2) \,:\, \dim V_\omega^2 = l\},\ 1 \leq l \leq d$$

is a measurable subset and for each l there exist measurable functions

$$v_1, \dots, v_l : \Omega_l^2 \to \mathbb{R}^d$$

such that $\{v_1(\omega), \dots, v_l(\omega)\}$ is a basis of V_ω^2 for every ω. Then, in particular

$$\lambda_2(\omega) = \max\{\lambda(\omega, v_i(\omega)) : 1 \leq i \leq l\}$$

is a measurable function on Ω_l^2 for every $1 \leq l \leq d$.

Repeating these arguments, we find that

(i) For every $c \geq 1$ the set $\{\omega \in \Omega \,:\, k(\omega) \geq c\}$ is measurable such that the map $\omega \to k(\omega)$ is measurable,

(ii) each Lyapunov exponent $\lambda_i(\omega)$ is a measurable function of ω on the set $\pi(V_*^i) = \{\omega \in \Omega : k(\omega) \geq i\}$,

(iii) each Oseledets space V_ω^i is a measurable function of ω on $\pi(V_*^i)$.

Hence, we are done with part (b) of the theorem.

[**End of Lecture VI, 18.05.**]

3. Finding the Lyapunov exponents as limits in Oseledets spaces: Part (c) turns out to be by far the hardest part of the proof. First of all we establish a statement about the time averages of skew products. Let X be a compact metric space. We introduce the space $\mathcal{C}(\Omega \times X)$ as the space of measurable observables $H : \Omega \times X \to \mathbb{R}$ such that $H(\omega, \cdot) \in C^0(X)$ for almost all ω and $\omega \mapsto \|H(\omega, \cdot)\|_\infty$ is integrable. We can then define the complete norm

$$\|H\|_1 = \int \|H(\omega, \cdot)\|_\infty \, \mathrm{d}\mathbb{P}(\omega) \tag{2.5.5}$$

on $\mathcal{C}(\Omega \times X)$. For continuous discrete-time RDS $(\Theta) : \Omega \times X \to \Omega \times X$, we can then state:

Proposition 2.5.8. *For any $H \in \mathcal{C}(\Omega \times X)$, define*

$$I(\omega) = \lim_{n\to\infty} \frac{1}{n} \inf_{v\in X} \sum_{j=0}^{n-1} H\left(\Theta^j(\omega, v)\right) \quad \text{and} \quad S(\omega) = \lim_{n\to\infty} \frac{1}{n} \sup_{v\in X} \sum_{j=0}^{n-1} H\left(\Theta^j(\omega, v)\right).$$

Then the limit exists at $\mathbb{P}$-almost every point and there are invariant measures μ_I and μ_S of Θ such that

$$\int H \, \mathrm{d}\mu_I = \int I \, \mathrm{d}\mathbb{P} \quad \text{and} \quad \int H \, \mathrm{d}\mu_S = \int S \, \mathrm{d}\mathbb{P}. \tag{2.5.6}$$

The proof can be done via the subadditive ergodic theorem and using Proposition 2.5.6. Similarly, by using Birkhoff's ergodic theorem and Proposition 2.5.5, one obtains the following corollary:

Corollary 2.5.9. *For almost all $\omega \in \Omega$ there are $v_I(\omega) \in X$ and $v_S(\omega) \in X$ such that*

$$I(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} H\left(\Theta^j(\omega, v_I(\omega))\right) \quad \text{and} \quad S(\omega) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} H\left(\Theta^j(\omega, v_S(\omega))\right).$$

3.1 Coinciding matrix limits in Oseledets spaces: We now apply these observations to linear cocycles. In more detail, we prove that for any measurable invariant subbundle $\omega \mapsto V_\omega$ for Θ, i.e. $V_\omega \subset \mathbb{R}^d$ is a linear subspace and $A(\omega)V_\omega = V_{\theta\omega}$, we have for almost all $\omega \in \Omega$

$$\lim_{n\to\infty} \frac{1}{n} \log \|(A^n(\omega)|V_\omega)^{-1}\|^{-1} = \min\{\lambda(\omega, v) \,:\, v \in V_\omega \setminus \{0\}\}; \tag{2.5.7}$$

$$\lim_{n\to\infty} \frac{1}{n} \log \|(A^n(\omega)|V_\omega)\| = \max\{\lambda(\omega, v) \,:\, v \in V_\omega \setminus \{0\}\}. \tag{2.5.8}$$

Taking $V_\omega = \mathbb{R}^d$, this immediately implies $\lambda_+(\omega) = \lambda_1(\omega)$ and $\lambda_-(\omega) = \lambda_{k(\omega)}$ for almost all $\omega \in \Omega$.

To prove (2.5.7) and (2.5.8), we may suppose that $\dim V_\omega = l$ for all ω, up to restricting to invariant subsets of Ω. Note that using Gram-Schmidt we can find measurable functions $\{v_1(\omega), \dots, v_l(\omega)\}$ that are an orthonormal basis of V_ω, identifying V_ω with $\mathbb{R}^l$ through an isometry.

We will denote $D(\omega) = A(\omega)|V_\omega$ and let $\Theta_D : \Omega \times \mathbb{R}^l \to \Omega \to \mathbb{R}^l$ be given by $\Theta_D(\omega, v) = (\theta\omega, D(\omega)v)$. We clearly may induce $\log^+ \|D^{\pm 1}\| \in L^1(\mathbb{P})$. Denoting by $\mathbb{P}\mathbb{R}^l$ the projective space with respect to $\mathbb{R}^l$, i.e. the space of vector lines through $\{0\}$ in $\mathbb{R}^l$, we introduce the projectivization of Θ_D

$$\tilde{\Theta}_D : \Omega \times \mathbb{PR}^l \to \Omega \times \mathbb{PR}^l.$$

Consider $H : \Omega \times \mathbb{PR}^l \to \mathbb{R}$, defined by

$$H(\omega, [v]) = \log \frac{\|D(\omega)v\|}{\|v\|}.$$

Note that clearly $H \in \mathcal{C}(\Omega \times \mathbb{PR}^l)$. For any $v \in \mathbb{R}^l \setminus \{0\}$ and any $n \geq 0$,

$$\begin{aligned}\limsup_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} H\left(\tilde{\Theta}_D^j(\omega, [v])\right) &= \limsup_{n\to\infty} \frac{1}{n} \log \prod_{j=0}^{n-1} \frac{\|D^{j+1}(\omega)v\|}{\|D^j(\omega)v\|} \\ &= \limsup_{n\to\infty} \frac{1}{n} \log \frac{\|D^n(\omega)v\|}{\|v\|} = \lambda^D(\omega, v),\end{aligned}$$

where $\lambda^D(\omega, v)$ denotes the corresponding exponents (2.5.4) for D. We observe that for every $n \geq 0$

$$I_n(\omega) := \inf_{[v]\in\mathbb{PR}^l} \sum_{j=0}^{n-1} H\left(\tilde{\Theta}_D^j(\omega, [v])\right) = \log \|D^n(\omega)^{-1}\|^{-1}$$

and

$$S_n(\omega) := \sup_{[v]\in\mathbb{PR}^l} \sum_{j=0}^{n-1} H\left(\tilde{\Theta}_D^j(\omega, [v]\right) = \log \|D^n(\omega)\|.$$

Hence, in the sense of Proposition 2.5.8, we have that

$$I(\omega) = \lim_{n\to\infty} \frac{1}{n} I_n(\omega) = \lambda_-^D(\omega) \;\; \text{and} \;\; S(\omega) = \lim_{n\to\infty} \frac{1}{n} S_n(\omega) = \lambda_+^D(\omega).$$

In particular, with Corollary 2.5.9, we deduce that

$$\min\{\lambda^D(\omega, v) \,:\, v \in \mathbb{R}^l \setminus \{0\}\} = \lambda_-^D(\omega), \; \max\{\lambda^D(\omega, v) \,:\, v \in \mathbb{R}^l \setminus \{0\}\} = \lambda_+^D(\omega),$$

which shows equations (2.5.7) and (2.5.8).

3.2: Dimension reduction: We will now reduce the problem to two-dimensional cocycles and then be able to wrap up the proof. For a measurable invariant subbundle $\omega \to V_\omega$, let $\alpha(\omega) < \beta(\omega)$ be measurable invariant functions such that almost surely

(i) $\lambda(\omega, v) \leq \alpha(\omega)$ for all $v \in V_\omega \setminus \{0\}$;

(ii) $\lambda(\omega, u) \geq \beta(\omega)$ for all $u \in \mathbb{R}^d \setminus V_\omega$.

By relation (2.5.8), we observe that

(iii) $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)|V_\omega\| \leq \alpha(\omega)$.

Let $V_\omega^\perp$ be the orthogonal complement of V_ω, where $\omega \to V_\omega^\perp$ is measurable since the orthogonal complement map $\perp : \mathrm{Gr}(l, d) \to \mathrm{Gr}(d - l, d)$ is a diffeomorphism for every l.

Note that we may write A, when taken relatively to the direct sum decomposition $\mathbb{R}^d = V_\omega^\perp \oplus V_\omega$, as

$$A(\omega) = \begin{pmatrix} B(\omega) & 0 \\ C(\omega) & D(\omega) \end{pmatrix}, \tag{2.5.9}$$

where again $D(\omega)$ is the restriction $A(\omega)|V_\omega$. Clearly, we may infer

$$\log^+ \|B^{\pm 1}\|,\ \log^+ \|C^{\pm 1}\|,\ \log^+ \|D^{\pm 1}\| \in L^1(\mathbb{P}).$$

The last two observations that we have to prove before putting together the full proof are the following: for almost all ω, any $u \in V_\omega^\perp \setminus \{0\}$ and any $v \in V_\omega$

(a) $\limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)(u+v)\|$,

(b) if $\lim_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\|$ exists, then $\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)(u+v)\|$ exists for all $v \in V_\omega$ and the two limits coincide.

Firstly, it is easy to observe (check!) that for any $u \in V_\omega^\perp \setminus \{0\}$ and any $v \in V_\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)(u+v)\|.$$

So from now on we just consider the problem for $v = 0$. We will further use the following fact:

Lemma 2.5.10. *For any $\varepsilon > 0$, there is a measurable function $d_\varepsilon(\omega) > 0$ such that*

$$\|D^n(\theta^m(\omega))\| \le d_\varepsilon e^{\alpha(\omega)n+(m+n)\varepsilon} \quad \textit{for all } m, n \ge 0. \tag{2.5.10}$$

Proof. See QS 7. The main idea is to define

$$1 \le b_\varepsilon(\omega) = \sup\{\|D^n\omega\| e^{-n(\alpha(\omega)+\varepsilon)} \ :\ n \ge 0\} < \infty$$

and to deduce that

$$d_\varepsilon(\omega) := \sup\{b_\varepsilon(\theta^m\omega)e^{-\varepsilon m} \ :\ m \ge 0\}$$

is finite almost surely. Then it is straightforward to derive the claim. □

Now observe that we may write, for every $n \ge 0$,

$$A^n(\omega) = \begin{pmatrix} B^n(\omega) & 0 \\ C_n(\omega) & D^n(\omega), \end{pmatrix}$$

where

$$C_n(\omega) = \sum_{j=0}^{n-1} D^{n-j-1}(\theta^{j+1}\omega) C(\theta^j\omega) B^j(\omega). \tag{2.5.11}$$

Fixing $\omega \in \Omega$ and $u \in V_\omega \setminus \{0\}$, we consider

$$\gamma = \max \left\{ \alpha(\omega), \limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| \right\}.$$

Hence, by definition, we have for any $\varepsilon > 0$ a real number b_ε such that

$$\|B^j(\omega)u\| \le b_\varepsilon e^{j(\gamma+\varepsilon)} \quad \text{for every } j. \tag{2.5.12}$$

Using the already often deployed subexponential growth of measurable observables whose logarithm is integrable, we observe that there is a measurable function c_ε such that

$$\|C(\theta^j\omega)\| \le c_\varepsilon(\omega) e^{j\varepsilon} \quad \text{for every } j. \tag{2.5.13}$$

Combining estimates (2.5.10), (2.5.12) and (2.5.13) in (2.5.11) we deduce

$$\|C_n(\omega)u\| \le \sum_{j=0}^{n-1} d_\varepsilon(\omega) e^{(n-j-1)\alpha(\omega)+n\varepsilon} c_\varepsilon(\omega) e^{j\varepsilon} b_\varepsilon e^{j(\gamma+\varepsilon)} \le n\alpha_\varepsilon e^{n(\gamma+3\varepsilon)},$$

where $\alpha_\varepsilon = b_\varepsilon c_\varepsilon(\omega) d_\varepsilon(\omega)$. This yields

$$\limsup_{n\to\infty} \frac{1}{n} \log \|C_n(\omega)u\| \le \gamma + 3\varepsilon.$$

Since $u \in V_\omega^\perp$, we have $A^n(\omega)u = (B^n(\omega)u, C_n(\omega)u)^\top$ and, in particular,

$$\|A^n(\omega)u\|^2 = \|B^n(\omega)u\|^2 + \|C_n(\omega)u\|^2.$$

Hence, we obtain, due to ε being arbitrary,

$$\limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| \le \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| \le \gamma. \tag{2.5.14}$$

By our definitions of α and β we infer

$$\alpha(\omega) < \beta(\omega) \le \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| \le \gamma.$$

and, hence, from the definition of γ we can now deduce that

$$\limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| = \gamma. \tag{2.5.15}$$

Thus, the relations (2.5.14) and (2.5.15) yield part (a) above.

Now to part (b). Note that

$$\|A^n(\omega)(u+v)\|^2 = \|B^n(\omega)u\|^2 + \|C_n(\omega)u + D^n(\omega)v\|^2,$$

and, hence, under the assumption of the limit existing,

$$\liminf_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)(u+v)\| \geq \liminf_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| = \limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\|.$$

Hence, the claim follows immediately from part (a).

3.3 Completion of the proof in discrete time: We will now use the previous observations to show that

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| \tag{2.5.16}$$

exists for almost all $\omega \in \Omega$ and every $v \in V^i_\omega \setminus V^{i+1}_\omega$, $1 \leq i \leq k$. Then, obviously, $\lambda(\omega, v) = \lambda_i(\omega)$.

Under replacing Ω by suitable invariant subsets, we may assume that $k(\omega) =: k$ is independent from ω, and so is the dimension $l \geq 1$ of the invariant subbundle $V_\omega := V^k_\omega$. Let $\alpha(\omega) = \lambda_k(\omega)$ and $\beta(\omega) = \lambda_{k-1}(\omega)$ such that we are in the situation of part **3.2** of the proof with (a) and (b) being true. Furthermore, observe that we can apply (2.5.8) and (2.5.7) to see

$$\lim_{n\to\infty} \frac{1}{n} \log \|(A^n(\omega)|V_\omega)^{-1}\|^{-1} = \lambda_k(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)|V_\omega\|$$

and, hence,

$$\lambda_k(\omega) = \lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)v\| \quad \text{for all } v \in V_\omega \setminus \{0\}. \tag{2.5.17}$$

Recall our expression for A with respect to the direct sum decomposition $\mathbb{R}^d = V^\perp_\omega \oplus V_\omega$. We may again identify $V^\perp_\omega$ with $\mathbb{R}^{d-l}$ and view each $B(\omega)$ as an element of $\mathrm{GL}(d-l)$. Define now for all $i = 1, \dots, k$

$$U^i_\omega = V^\perp_\omega \cap V^i_\omega.$$

Then by property (a) above, we have for all $1 \leq i \leq k-1$ and $u \in U^i_\omega \setminus U^{i+1}_\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| = \lambda_i(\omega).$$

Thus, $\mathbb{R}^{d-l} = U^1_\omega \supsetneq \cdot \supsetneq U^{k-1}_\omega \supsetneq \{0\}$ is the Oseledets flag of B with Lyapunov exponents $\lambda_1(\omega), \dots, \lambda_{k-1}(\omega)$. Hence, we can now derive the limit for λ_{k_1} analagously as in (2.5.17), and by induction obtain

$$\lim_{n\to\infty} \frac{1}{n} \log \|B^n(\omega)u\| = \lambda_i(\omega) \quad \text{for all } u \in U^i_\omega \setminus U^{i+1}_\omega$$

and every $i = 1, \dots, k-1$. Hence, by (b) above, we can deduce the final statement.

4 Transition to continuous time: For reasons of time and space, we refrain from giving detailed arguments here and refer to [2, Section 3.4] and, even more specifically, [24, Chapter 5, §2]. The main observation is that one may use the cocycle property to observe for any $t \in \mathbb{R}^+$ with $t = n + s$, $n \in \mathbb{N}$ and $s \in [0,1]$,

$$\Phi(t,\omega) = \Phi(n+s,\omega) = \Phi(n,\theta_s\omega)\Phi(s,\omega) = \Phi(s,\theta_n\omega)\Phi(n,\omega),$$

such that, under condition (2.5.2), the statements can be inferred from discrete time relatively straight-forwardly. □

We can now immediately go to finishing the proof of the MET in two-sided time, i.e. upgrading the Oseledets flag to a decomposition and proving the subexponential decay of the angles.
[**End of Lecture VII, 25.05.**]

Proof of Theorem 2.5.3. In the following, we will again take without loss of generality $k(\omega)$ and dimension l of $V_\omega := V_\omega^k$ to be constant in ω, and set $\alpha(\omega) = \lambda_k(\omega)$ and $\beta(\omega) = \lambda_{k-1}(\omega)$.

We again write A as a lower triangular matrix (2.5.9) with respect to $\mathbb{R}^d = V_\omega^\perp \oplus V_\omega$, and get, now also in two-sided time,

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|D^n(\omega)^{-1}\|^{-1} = \lim_{n\to\pm\infty} \frac{1}{n} \log \|D^n(\omega)\| = \alpha(\omega) \tag{2.5.18}$$

for almost all ω. From the proof of Theorem 2.5.2 we observe that for almost all ω

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|B^n(\omega)^{-1}\|^{-1} = \beta(\omega) \tag{2.5.19}$$

We can use these facts to show the following:

Proposition 2.5.11. *If $\theta : \Omega \to \Omega$ is invertible, there exists a measurable invariant subbundle $\omega \to W_\omega$ such that $\mathbb{R}^d = W_\omega \oplus V_\omega$ for almost all ω.*

Proof. Let $\mathcal{L}$ be the space of measurable maps $L : \omega \to L_\omega$ where $L_\omega : V_\omega^\perp \to V_\omega$ is a linear map. We introduce the graph transform $T : \mathcal{L} \to \mathcal{L}$ as the transformation characterized by the condition that for all $\omega \in \Omega$ the image of the graph of L_ω under $A(\omega)$ coincides with the graph of $T(L)_{\theta\omega}$, i.e.

$$A(\omega) \begin{pmatrix} v \\ L_\omega v \end{pmatrix} = \begin{pmatrix} w \\ T(L)_{\theta\omega} w \end{pmatrix}.$$

In particular this yields the relation

$$T(L)_{\theta\omega} = [C(\omega) + D(\omega)L_\omega]B(\omega)^{-1} \quad \text{for all } \omega. \tag{2.5.20}$$

Note that the graph of some $L \in \mathcal{L}$ gives an invariant subbundle if and only if $T(L)_\omega = L_\omega$ for almost all ω. Hence, proving the proposition amounts to finding a fixed point of the graph transform.

We rewrite (2.5.20) as

$$T(L)_{\tilde{\omega}} = R_{\tilde{\omega}} + S(L)_{\tilde{\omega}} \quad \text{for all } \tilde{\omega}, \tag{2.5.21}$$

where

$$R : \tilde{\omega} \mapsto R_{\tilde{\omega}} = C(\theta^{-1}\tilde{\omega})B(\theta^{-1}\tilde{\omega})^{-1}$$

is an element of $\mathcal{L}$ and $S : \mathcal{L} \to \mathcal{L}$ is the linear operator defined by

$$S(L)_{\tilde{\omega}} = D(\theta^{-1}\tilde{\omega})L_{\theta^{-1}\tilde{\omega}}B(\theta^{-1}\tilde{\omega})^{-1}.$$

Claim: There exist measurable functions $\alpha(\omega) > 0$ and $\varepsilon(\omega) > 0$ such that

$$\|S^k(R)_{\tilde{\omega}}\| \leq a(\tilde{\omega})e^{-k\varepsilon(\tilde{\omega})} \quad \text{for every } k \geq 0 \quad \text{and almost all } \tilde{\omega}. \tag{2.5.22}$$

Assuming this for now, we may define

$$L : \tilde{\omega} \mapsto L_{\tilde{\omega}} = \sum_{k=0}^{\infty} S^k(R)_{\tilde{\omega}}$$

as an element of $\mathcal{L}$. In particular, observe that

$$T(L)_{\tilde{\omega}} = R_{\tilde{\omega}} + S(L)_{\tilde{\omega}} = R_{\tilde{\omega}} + S\left(\sum_{k=0}^{\infty} S^k(R)\right)_{\tilde{\omega}} = R_{\tilde{\omega}} + \sum_{k=1}^{\infty} S^k(R)_{\tilde{\omega}} = \sum_{k=0}^{\infty} S^k(R)_{\tilde{\omega}} = L_{\tilde{\omega}}.$$

Hence, it only remains to prove the claim (2.5.22), which has been done on QS 7, Exercise 2, using Lemma 2.5.10 in combination with (2.5.18) and (2.5.19). □

Observe that, by relation (2.5.18),

$$\lambda_k(\omega) = \alpha(\omega) = \lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(\omega)v\| \quad \text{for all } v \in V_\omega \setminus \{0\}. \tag{2.5.23}$$

Using the decomposition $\mathbb{R}^d = W_\omega \oplus V_\omega$, we may identify W_ω with $\mathbb{R}^{d-l}$ and let $\tilde{A} : \Omega \to \mathrm{GL}(d-l)$ be given by $\tilde{A}(\omega) = A(\omega)|W_\omega$. Define now for all $i = 1, \ldots, k$

$$W^i_\omega = W_\omega \cap V^i_\omega.$$

Then by property (a) from the proof of Theorem 2.5.2, we have for all $1 \leq i \leq k-1$ and $u \in W^i_\omega \setminus W^{i+1}_\omega$

$$\limsup_{n\to\infty} \frac{1}{n} \log \|\tilde{A}^n(\omega)u\| = \limsup_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| = \lambda_i(\omega).$$

Thus, $\mathbb{R}^{d-l} = W^1_\omega \supsetneq \cdot \supsetneq W^{k-1}_\omega \supsetneq \{0\}$ is the Oseledets flag of $\tilde{A}$ with Lyapunov exponents $\lambda_1(\omega), \ldots, \lambda_{k-1}(\omega)$. Hence, we can now derive the limit for λ_{k-1} analagously as in (2.5.23), and by induction obtain an $\tilde{A}$-invariant splitting $W_\omega = E^1(\omega) \oplus \cdots \oplus E^{k-1}(\omega)$ such that

$$W^j_\omega = \bigoplus_{i=j}^{k-1} E^i(\omega) \quad \text{for every } j = 1, \ldots, k-1,$$

$$\lim_{n\to\infty} \frac{1}{n} \log \|A^n(\omega)u\| = \lambda_i(\omega) \quad \text{for all } u \in E^i(\omega).$$

Denoting $E^k(\omega) = V_\omega = V_\omega^k$, we obtain the A-invariant splitting

$$\mathbb{R}^d = \bigoplus_{i=1}^{k} E^i(\omega)$$

with

$$V_\omega^j = V_\omega^k \oplus W_\omega^j = \bigoplus_{i=j}^{k} E^i(\omega) \quad \text{for every } j = 1, \dots, k-1,$$

This proves part (a) and (b).

It remains to show the subexponential decay of angles (c). Given disjoint subsets I and J of $\{1, \dots, k\}$, define

$$\phi(\omega) = \left| \sin \measuredangle \left(\bigoplus_{i \in I} E^i(\theta^n \omega), \bigoplus_{j \in J} E^j(\theta^n \omega) \right) \right|.$$

Due to the invariance of the Oseledets subspaces, we obtain

$$\frac{\phi(\theta\omega)}{\phi(\omega)} = \frac{\left| \sin \measuredangle \left(A(\omega) \bigoplus_{i \in I} E^i(\theta^n \omega), A(\omega) \bigoplus_{j \in J} E^j(\theta^n \omega) \right) \right|}{\left| \sin \measuredangle \left(\bigoplus_{i \in I} E^i(\theta^n \omega), \bigoplus_{j \in J} E^j(\theta^n \omega) \right) \right|}.$$

Similarly to before one can observe by elementary linear algebra (check!) that

$$(\|A(\omega)\| \|A(\omega)^{-1}\|)^{-1} \leq \frac{\left| \sin \measuredangle \left(A(\omega) \bigoplus_{i \in I} E^i(\theta^n \omega), A(\omega) \bigoplus_{j \in J} E^j(\theta^n \omega) \right) \right|}{\left| \sin \measuredangle \left(\bigoplus_{i \in I} E^i(\theta^n \omega), \bigoplus_{j \in J} E^j(\theta^n \omega) \right) \right|} \leq \|A(\omega)\| \|A(\omega)^{-1}\|.$$

Hence, we obtain that $\log \phi \circ \theta - \log \phi$ is $\mathbb{P}$-integrable, and equivalently, that $\log \phi \circ \theta^{-1} - \log \phi$ is $\mathbb{P}$-integrable. Hence, applying our well-known statement about sublinear growth of integrable observables we get

$$\lim_{n \to \pm\infty} \frac{1}{n} \phi(\theta^n \omega) = 0,$$

which finishes the proof for discrete time.

The transition to the time-continuous case is as before. □

We add a couple of remarks to Theorems 2.5.2 and 2.5.3.

Remark 2.5.12. (a) Note that one can write the discrete time part of Theorem 2.5.2 also for non-invertible matrix generators $A(\omega)$, i.e. the main condition reduces to

$$\log^+ \|A(\cdot)\| \in L^1(\mathbb{P}).$$

Then all the statements hold for a forward invariant set $\tilde{\Omega} \subset \Omega$ of full measure, allowing for $\lambda_{k(\omega)} = -\infty$ as a well-defined limit by considering $\mathbb{R} \cup \{-\infty\}$. The Oseledets subspaces are then only forward-invariant, i.e.

$$A(\omega)V_i(\omega) \subset V_i(\theta\omega)$$

for all $i = 1, \dots, k(\omega)$.

(b) Recall from Theorem 2.5.2 that, if the underlying measure $\mathbb{P}$ is ergodic, then the number of spaces $k(\omega)$ and the dimension of V_ω^i, for all $i = 1, \dots, k$, is constant for almost all ω. Hence, we find for fixed $\omega \in \tilde{\Omega}$, where $\tilde{\Omega}$ is the invariant set of full measure for which the MET holds, and an Oseledets flag space V_ω^i with dimension l, a measurable basis $\{v_1(\omega), \dots, v_l(\omega)\}$ of V_ω^i (also cf. QS 6, Exercise 1). Setting $v_j(\theta\omega) = A(\omega)v_j(\omega) \in V_{\theta\omega}^i$, we observe that

$$0 = \sum_{j=1}^{l} \lambda_j v_j(\theta\omega) = A(\omega) \sum_{j=1}^{l} \lambda_j v_j(\omega)$$

implies $\lambda_1 = \dots = \lambda_l = 0$. Hence, $\{v_1(\theta\omega), \dots, v_l(\theta\omega)\}$ is a measurable basis of $V_{\theta\omega}^i$, such that we obtain an invariant basis for each Oseledets subspace on $\tilde{\Omega}$.

(c) Note that in the proof of the two-sided theorem 2.5.3, we actually showed

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \|(A^n(\omega)|E^i(\omega))^{-1}\|^{-1} = \lim_{n\to\pm\infty} \frac{1}{n} \log \|A^n(\omega)|E(\omega)^i\| = \lambda_i(\omega).$$

From this, we obtain, by bounding the determinant from above and below by the corresponding powers of the matrix norm, that

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left|\det\left(A^n(\omega)|E^i(\omega)\right)\right| = \lambda_i(\omega) \dim E^i(\omega)$$

almost surely. Using Theorem 2.5.3 (c), i.e. the subexponential decay of angles between subspaces, we obtain

$$\lim_{n\to\pm\infty} \frac{1}{n} \log \left|\det\left(A^n(\omega)| \bigoplus_{i\in I} E^i(\omega)\right)\right| = \sum_{i\in I} \lambda_i(\omega) \dim E^i(\omega)$$

for any $I \subset \{1, \dots, k(\omega)\}$. In particular, we obtain

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(A^n(\omega)| = \sum_{i=1}^{k(\omega)} \lambda_i(\omega) \dim E^i(\omega), \tag{2.5.24}$$

and the formulas

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(A^n(\omega)|E^u(\omega))| = \sum_{\lambda_i(\omega)>0} \lambda_i(\omega) \dim E^i(\omega) \tag{2.5.25}$$

$$\lim_{n\to\pm\infty} \frac{1}{n} \log |\det(A^n(\omega)|E^s(\omega))| = \sum_{\lambda_i(\omega)<0} \lambda_i(\omega) \dim E^i(\omega) \tag{2.5.26}$$

where $E^s(\omega)) = \bigoplus_{\lambda_i(\omega)<0} E^i(\omega)$ is the *stable* and $E^u(\omega)) = \bigoplus_{\lambda_i(\omega)>0} E^i(\omega)$ the unstable bundle.

(d) Note that there are various notions of the Multiplicative Ergodic Theorem, that extend our result for Euclidean state space $\mathbb{R}^d$ to general smooth manifolds, and also infinite-dimensional spaces, i.e. Hilbert and Banach spaces. Extending such formulations is still part of active research and an interesting field to engage in.

Note that the following two subsections have not been discussed in the lecture and are only there the convenience of the reader.

2.5.1 The MET for differentiable continuos-time RDS

The background given in Section 2.1 plus the MET given in the previous section lead to the following formulation of the MET, which can be applied to differentiable RDS induced by SDEs with unique, and therefore ergodic, invariant Markov measure. The proof directly follows from the previous parts of this chapter:

Theorem 2.5.13 (Multiplicative Ergodic Theorem for derivative cocycle)**.** *Suppose the C^1-random dynamical system (θ, φ) on $\mathbb{R}^d$, defined in one-sided time, has an ergodic invariant measure ν and satisfies the integrability condition*

$$\sup_{0 \leq t \leq 1} \log^+ \|\mathrm{D}\varphi(t, \omega, x)^{\pm 1}\| \in L^1(\nu).$$

(a) Then there exist a Θ-invariant set $\Delta \subset \Omega \times X$ with $\nu(\Delta) = 1$, a number $1 \leq k \leq d$ and real numbers $\lambda_1 > \cdots > \lambda_k$, the Lyapunov exponents with respect to ν, such that for all $0 \neq v \in T_x X \cong \mathbb{R}^d$ and $(\omega, x) \in \Delta$

$$\lambda(\omega, x, v) := \lim_{t \to \infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, x) v\| \in \{\lambda_1, \ldots, \lambda_k\}.$$

Furthermore, the tangent space $T_x X \cong \mathbb{R}^d$ admits a flag

$$\mathbb{R}^d = V_1(\omega, x) \supsetneq V_2(\omega, x) \supsetneq \cdots \supsetneq V_k(\omega, x) \supsetneq V_{k+1}(\omega, x) = \{0\},$$

for all $(\omega, x) \in \Delta$ such that

$$\lambda(\omega, x, v) = \lambda_i \quad \iff \quad v \in V_i(\omega, x) \setminus V_{i+1}(\omega, x) \quad \text{for all } i \in \{1, \ldots, k\}.$$

In particular, we have for all $(\omega, x) \in \Delta$

$$\lim_{t \to \infty} \frac{1}{t} \ln \det |\mathrm{D}\varphi(t, \omega, x)| = \sum_{i=1}^{k} d_i \lambda_i, \qquad (2.5.27)$$

where d_i is the multiplicity of the Lyapunov exponent λ_i and $\sum_{i=1}^{p} d_i = d$.

(b) If the cocycle φ is defined in two-sided time and satisfies the above integrability condition also in backwards time, there exists the Oseledets splitting

$$\mathbb{R}^d = E_1(\omega, x) \oplus \cdots \oplus E_k(\omega, x)$$

of the tangent space into random subspaces $E_i(\omega, x)$, the Oseledets spaces, for all $(\omega, x) \in \Delta$. These have the following properties for all $(\omega, x) \in \Delta$:

(i) The Oseledets spaces are invariant under the derivative flow, i.e. for all $t \in \mathbb{R}$

$$\mathrm{D}\varphi(t, \omega, x) E_i(\omega, x) = E_i(\Theta_t(\omega, x)),$$

(ii)

$$\lim_{t \to \pm\infty} \frac{1}{t} \ln \|\mathrm{D}\varphi(t, \omega, x) v\| = \lambda_i \iff v \in E_i(\omega, x) \setminus \{0\} \quad \textit{for all } i \in \{1, \dots, p\},$$

(iii) The dimension equals the multiplicity of the associated Lyapunov exponent, i.e.

$$\dim E_i(\omega, x) = d_i .$$

2.5.2 The Furstenberg-Khasminskii formula

The standard method for deriving an explicit formula of the largest Lyapunov exponent λ_1 for RDS coming from SDEs is given by the *Furstenberg-Khasminskii formula*, which is developed on QS 7, based on [16].

Consider the linear Stratonovich equation

$$\mathrm{d}Y_t = A_0 Y_t \,\mathrm{d}t + \sum_{j=1}^{m} A_j Y_t \circ \mathrm{d}W_t^j , \quad Y_0 = v \in \mathbb{R}^d , \tag{2.5.28}$$

where $A_0, \dots, A_m \in \mathbb{R}^{d \times d}$ and $W^1, \dots, W^m$ are independent Wiener processes in two-sided time. For keeping things simple, we let $A_0, \dots, A_m \in \mathbb{R}^{d \times d}$ not depend on an underlying random system. Thus, equation (2.5.28) induces a linear cocycle Φ over the family of shifts $(\theta_t)_{t \in \mathbb{R}}$ on the Wiener space Ω.

We introduce the change of variables $r_t = \|Y_t\|$ and $s_t = Y_t / r_t$, so that s_t lies on the unit sphere $\mathbb{S}^{d-1}$. The stochastic differential equation in polar coordinates is given by

$$\mathrm{d}s_t = (A_0 s_t - \langle s_t, A_0 s_t \rangle s_t) \,\mathrm{d}t + \sum_{j=1}^{m} (A_j s_t - \langle s_t, A_j s_t \rangle s_t) \circ \mathrm{d}W_t^j ,$$

and

$$\mathrm{d}r_t = \langle s_t, A_0 s_t \rangle r_t \,\mathrm{d}t + \sum_{j=1}^{m} \langle s_t, A_j s_t \rangle r_t \circ \mathrm{d}W_t^j ,$$

We define

$$g_A(s) = As - \langle s, As \rangle s \quad \text{for } A \in \mathbb{R}^{d \times d}, s \in \mathbb{S}^{d-1}\,,$$

and denote by $\mathfrak{L}(g_{A_0}, \dots, g_{A_m})(s)$ the Lie algebra generated by these vector fields at s. We impose the classical Hörmander condition on the hypoellpticity of these vector fields driving the dynamics of s_t:

$$\dim \mathfrak{L}(g_{A_0}, \dots, g_{A_m})(s) = d - 1 \;\text{ for all } s \in \mathbb{S}^{d-1}\,. \tag{2.5.29}$$

According to [15], condition 2.5.29 guarantees that the distribution of the Oseledets space $E_i(\omega)$ possesses a smooth density for any $i \in \{1, \dots, p\}$. The hypoellipticity condition (2.5.29) further implies irreducibility of the Markov semigroup induced by $(s_t)_{t \geq 0}$ on $\mathbb{S}^{d-1}$, and since the unit sphere is a compact manifold, we can conclude that $(s_t)_{t \geq 0}$ possesses a unique stationary probability measure with smooth density p. The density p solves the stationary Fokker-Planck equation

$$\mathcal{L}^* p = 0\,,$$

where

$$\mathcal{L} = g_{A_0} + \frac{1}{2} \sum_{j=1}^m g_{A_j}^2$$

is the generator of $(s_t)_{t \geq 0}$ in Hörmander notation and $\mathcal{L}^*$ is the formal adjoint of $\mathcal{L}$. On QS 7 these insights are used to derive the the *Furstenberg–Khasminskii formula* for the top Lyapunov exponent, given by

$$\lambda_1 = \int_{\mathbb{S}^{d-1}} \left[h_{A_0}(s) + \sum_{j=1}^m k_{A_j}(s) \right] p(s)\,\mathrm{d}s\,, \tag{2.5.30}$$

where

$$\begin{aligned} h_A(s) &= \langle s, As \rangle\,, \\ k_A(s) &= \frac{1}{2} \langle (A + A^*)s, As \rangle - \langle s, As \rangle^2\,. \end{aligned}$$

2.6 Stable manifold theorem

We conclude this chapter with a statement about stable manifolds. In the literature, there are several distinctions on sufficient conditions for such a theorem to hold. We will not deal with these subtleties here but may come back to these when needed later.

We assume that our RDS (θ, φ) on $\mathbb{R}^d$ is C^2 and that all derivatives satisfy the integrability condition of the Multiplicative Ergodic Theorem with respect to an invariant measure ν. For simplicity we will assume that ν is ergodic (but the statements could again be also stated without ergodicity, with the corresponding dependencies on (ω, x)). An example would be the situation of Theorem 1.4.5 for $k = 2$, with unique stationary measure ρ such that $\nu = \mathbb{P} \times \rho$, when regarded in one-sided forward time as suitable for a stable manifold theorem (see Remark 1.5.7). Again, one can formulate the MET for the derivative cocycle in continuous time, where the proof is classically based on the discrete time version for iterates of time-one maps, which in the case of Theorem 1.4.5 are the C^2 random diffeomorphisms $\varphi(n, \omega, x)$ and their derivatives. This is also the way one typically proves the following theorem, deriving first an intricate local stable manifold theorem that we omit here for brevity.

Theorem 2.6.1 (Global stable manifold theorem for C^2 RDS)**.** *Take a C^2 RDS $\Theta = (\theta, \varphi)$ on $\mathbb{R}^d$ with ergodic invariant measure ν and assume that all derivatives satisfy the integrability condition of the Multiplicative Ergodic Theorem 2.5.2 (either in discrete or continuous time $t \in \mathbb{T}$). Let $\lambda_{s_1} > \cdots > \lambda_{s_p}$ be the strictly negative Lyapunov exponents and define $W^{s_p}(\omega, x) \subset \cdots \subset W^{s_1}(\omega, x)$ by*

$$W^{s_i}(\omega, x) = \left\{ y \in \mathbb{R}^d \, : \, \limsup_{t \to \infty} \frac{1}{t} \log \|\varphi(t, \omega, x) - \varphi(t, \omega, y)\| \leq \lambda_{s_i} \right\}, \ 1 \leq i \leq p.$$

(a) Then for ν-almost all (ω, x), we have that $W^{s_i}(\omega, x)$ is the image of the Oseledets subspace $V^{s_i}(\omega, x)$ under an injective immersion of class $C^{1,1}$ and is tangent to $V^{s_i}(\omega, x)$ at x.

(b) If $y \in W^{s_i}(\omega, x)$, then

$$\limsup_{t \to \infty} \frac{1}{t} \log d^s(\varphi(t, \omega, x), \varphi(t, \omega, y)) \leq \lambda_{s_i},$$

where d^s denotes the distance along the submanifold $\varphi(t, \omega, \cdot) W^{s_i}(\omega, x) \subset W^{s_i}(\theta_t \omega, \varphi(t, \omega, x))$.

(c) In particular, we have that the global stable manifold

$$W^s(\omega, x) = \left\{ y \in \mathbb{R}^d \, : \, \limsup_{t \to \infty} \frac{1}{t} \log \|\varphi(t, \omega, x) - \varphi(t, \omega, y)\| < 0 \right\}$$

satisfies ν-almost surely

$$W^s(\omega, x) = W^{s_1}(\omega, x),$$

and hence is the image of the Oseledets subspace $V^{s_1}(\omega, x)$ under an injective immersion of class $C^{1,1}$ and is tangent to $V^{s_1}(\omega, x)$ at x.

Proof. See [24, Theorem III, 3.2 and Theorem V, 2.2]. There the theorems are formulated on compact manifolds but this imposes no restrictions for us, as shown by Biskamp [6]. □

[**End of Lecture VIII, 01.06.**]

Chapter 3

Random attractors

3.1 Basic definitions, examples and an existence result

Let (θ, φ) be a random dynamical system on a Polish, i.e. complete and separable metric, space (X, d). Due to the non-autonomous nature of the RDS, there are no fixed attractors for dissipative systems and different notions of a random attractor exist. We introduce these related but different definitions of random attractors in the following, with respect to tempered sets. A random variable $R : \Omega \to \mathbb{R}^+$ is called *tempered* if

$$\lim_{t \to \pm\infty} \frac{1}{|t|} \log R(\theta_t \omega) = 0 \quad \text{for almost all } \omega \in \Omega\,,$$

see also [2, p. 164].

Definition 3.1.1. A set $D \in \mathcal{F} \otimes \mathcal{B}(X)$ is called a random set. The ω-section of a random set D is defined by

$$D(\omega) = \{x \,:\, (x, \omega) \in D\},\ \omega \in \Omega.$$

The set is called *tempered* if there exist a tempered random variable R and a point $x_0 \in X$ such that

$$D(\omega) \subset B_{R(\omega)}(x_0) \quad \text{for almost all } \omega \in \Omega\,,$$

where $B_{R(\omega)}(x_0)$ denotes a ball centered at x_0 with radius $R(\omega)$. If X is a vector space, we typically choose $x_0 = 0$, when not further specified.

Note that in particular all deterministic bounded sets are tempered.

D is called a compact (or closed) random set if $D(\omega) \subset X$ is compact (or closed) for almost all $\omega \in \Omega$. In this case, measurability of D amounts to measurability of the mapping $\omega \mapsto \inf_{y \in D(\omega)} d(x, y)$ for every $x \in X$.

Denote by $\mathcal{D}$ the set of all tempered sets $D \in \mathcal{F} \otimes \mathcal{B}(X)$ and by

$$\operatorname{dist}(E, F) := \sup_{x \in E} \inf_{y \in F} d(x, y)$$

the *Hausdorff seperation* or *semi-distance*. We now define different notions of a random attractor with respect to a family of sets $\mathcal{S} \subset \mathcal{D}$, see also [20, Definition 14.3] and [9, Definition 15].

Definition 3.1.2 (Random attractor)**.** Let $\mathcal{S} \subset \mathcal{D}$ be a class of tempered sets. A compact random set $A \in \mathcal{D}$ that is strictly φ-invariant, i.e.

$$\varphi(t,\omega)A(\omega) = A(\theta_t\omega) \quad \text{for all } t \geq 0 \text{ and almost all } \omega \in \Omega\,,$$

is called

(i) a *random pullback attractor* with respect to $\mathcal{S}$ if for all $D \in \mathcal{S}$ we have

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi(t,\theta_{-t}\omega)D(\theta_{-t}\omega), A(\omega)\big) = 0 \quad \text{for almost all } \omega \in \Omega\,,$$

(ii) a *random forward attractor* with respect to $\mathcal{S}$ if for all $D \in \mathcal{S}$ we have

$$\lim_{t\to\infty} \operatorname{dist}\big(\varphi(t,\omega)D(\omega), A(\theta_t\omega)\big) = 0 \quad \text{for almost all } \omega \in \Omega\,,$$

(iii) a *weak random attractor* if it satisfies the convergence property in (i) (or (ii)) with almost sure convergence replaced by convergence in probability,

(iv) a *(weak) random (pullback or forward) point attractor* if it satisfies the corresponding properties above for $\mathcal{S} = \{D \subset X : D = \{y\} \text{ for some } y \in X\}$, i.e. for single points $y \in X$.

(v) a minimal random attractor of any of the above kinds, if it is contained in any random attractor of its kind.

Note that due to the $\mathbb{P}$-invariance of θ_t for all $t \in \mathbb{R}$, it is easy to derive that weak attraction in the pullback and the forward sense are the same and, hence, the notion of a weak random attractor in Definition 3.1.2 (iii) is consistent (see exercises).

However, random pullback attractors and random forward attractors with almost sure convergence, as defined above, are generally not equivalent (see [25] for counter-examples). In the following, we will be careful with this distinction, yet in our main examples the random pullback attractor and random forward attractor will be the same. In this case we will simply speak of *the random attractor*.

Before we look at some examples, we add some remarks on Definition 3.1.2.

Remark 3.1.3. Note that we require that the random attractor is measurable with respect to $\mathcal{F} \otimes \mathcal{B}(X)$, in contrast to a weaker statement often used in the literature (see also [9, Remark 4]).

Remark 3.1.4. In many cases, the family of sets $\mathcal{S}$ is chosen to be the family of all bounded or compact (deterministic) subsets $B \subset X$, as for example in [14]. Note that our definition of random attractors is a generalization of this weaker definition.

Example 3.1.5. *1. Consider the already often used example of a pitchfork bifurcation with additive noise*

$$\mathrm{d}X_t = \left(\alpha X_t - X_t^3\right)\mathrm{d}t + \sigma \mathrm{d}W_t. \tag{3.1.1}$$

Then you will see on Question sheet 8 that the random attractor A is a singleton almost surely with respect to $\mathcal{S} = \mathcal{D}$, i.e. $A(\omega) = \{a(\omega)\}$ for a stationary solution $a(\omega)$ of the SDE (3.1.1).

2. Consider now the same example with linear multiplicative noise

$$\mathrm{d}X_t = \left(\alpha X_t - X_t^3\right)\mathrm{d}t + \sigma X_t \circ \mathrm{d}W_t. \tag{3.1.2}$$

Then one can observe that the cocycle solving the SDE (3.1.2) *is given by*

$$\varphi(t,\omega,x) = \frac{x\mathrm{e}^{\alpha t + \sigma W_t(\omega)}}{\left(1 + 2x^2 \int_0^t e^{2(\alpha s + \sigma W_s(\omega))}\mathrm{d}s\right)^{1/2}} \tag{3.1.3}$$

for all $x \in \mathbb{R}$. We have two cases:

(i) For $\alpha \le 0$, the random attractor is given by $A(\omega) = \{0\}$ for all $\omega \in \Omega$, with respect to $\mathcal{S} = \mathcal{D}$.

(ii) For $\alpha > 0$, the attractor for all tempered subsets (e.g. with respect to $x_0 = 1$) of $(0,\infty)$ is $A(\omega) = \{a(\omega)\}$ for

$$a(\omega) = \left(2\int_{-\infty}^0 e^{2(\alpha s + \sigma W_s(\omega))}\right)^{-1/2}$$

and for all tempered subsets of $(-\infty, 0)$ is $A(\omega) = \{-a(\omega)\}$.

3. Consider the planar stochastic differential equation

$$\begin{aligned} \mathrm{d}x &= \left(x - y - x\left(x^2+y^2\right)\right)\,\mathrm{d}t + \sigma x \circ \mathrm{d}W_t\,, \\ \mathrm{d}y &= \left(x + y - y\left(x^2+y^2\right)\right)\,\mathrm{d}t + \sigma y \circ \mathrm{d}W_t\,. \end{aligned} \tag{3.1.4}$$

where W_t denotes a one-dimensional standard Brownian motion and the noise is of Stratonovich type. We denote the cocycle of the induced random dynamical system by $\varphi = (\varphi_1, \varphi_2)$. Equation (3.1.4) *can be transformed into polar coordinates $(\vartheta, r) \in [0, 2\pi) \times [0,\infty)$*

$$\begin{aligned} \mathrm{d}\vartheta &= 1\,\mathrm{d}t, \\ \mathrm{d}r &= (r - r^3)\,\mathrm{d}t + \sigma r \circ \mathrm{d}W_t\,. \end{aligned} \tag{3.1.5}$$

Similarly to before, equation (3.1.5) *has an explicit unique solution given by*

$$\hat{\varphi}(t,\omega,(\vartheta_0,r_0)) = \left(\vartheta_0 + t \mod 2\pi,\ \frac{r_0 \mathrm{e}^{t+\sigma W_t(\omega)}}{\left(1 + 2r_0^2 \int_0^t \mathrm{e}^{2(s+\sigma W_s(\omega))} \mathrm{d}s\right)^{1/2}}\right)$$
$$=: (\vartheta(t,\omega,\vartheta_0), r(t,\omega,r_0))\,.$$

Moreover, there is a stationary solution for the radial component, satisfying $r(t,\omega,r^*(\omega)) = r^*(\theta_t\omega)$*, and given by*

$$r^*(\omega) = \left(2\int_{-\infty}^0 \mathrm{e}^{2s+2\sigma W_s(\omega)} \mathrm{d}s\right)^{-1/2}. \tag{3.1.6}$$

In particular, one can see from a straightforward computation that for all $(x,y) \neq (0,0)$ *and almost all* $\omega \in \Omega$

$$\left(\varphi_1(t,\theta_{-t}\omega,x)^2 + \varphi_2(t,\theta_{-t}\omega,y)^2\right)^{1/2} \to r^*(\omega) \quad \textit{as } t \to \infty\,,$$

and also

$$\left(\varphi_1(t,\omega,x)^2 + \varphi_2(t,\omega,y)^2\right)^{1/2} \to r^*(\theta_t\omega) \quad \textit{as } t \to \infty\,.$$

Hence, the planar system (3.1.4) *has a random attractor* A *in the pullback and forward sense, with respect to* $\mathcal{S} = \mathcal{D} \setminus \{\{0\}\}$*, where* $\mathcal{D}$ *denotes the set of all compact tempered sets* $D \in \mathcal{F} \otimes \mathcal{B}(\mathbb{R}^2)$*, and the fibers of* A *are given by (see Figure 3.1)*

$$A(\omega) = \{r^*(\omega)(\cos\alpha, \sin\alpha) \,:\, \alpha \in [0,2\pi)\}. \tag{3.1.7}$$

We derive a criterion for the existence of random attractors via Ω-limit sets and absorbing and attracting sets. In more details, let us assume continuity of our RDS in the following and introduce the following definition:

Definition 3.1.6. Given a random set K, the set

$$\Omega_K(\omega) = \bigcap_{\tau \geq 0} \overline{\bigcup_{t \geq \tau} \varphi(t,\theta_{-t}\omega)K(\theta_{-t}\omega)}$$

is called the Ω-limit set of K. Clearly, $\Omega_K(\omega)$ is closed.

Note that we may identify, in analogy to deterministic dynamical systems,

$$\Omega_K(\omega) = \{y \in X \,:\, \text{there are } t_n \to \infty, x_n \in K(\theta_{-t_n}\omega) \text{ s.t. } \varphi(t_n,\theta_{-t_n}\omega,x_n) \to y\}.$$

One may observe the following:

Lemma 3.1.7. *The* Ω*-limit set of an arbitrary random set* K *is forward-invariant, i.e. for all* $t \geq 0$

$$\varphi(t,\omega)\Omega_K(\omega) \subset \Omega_K(\theta_t\omega).$$

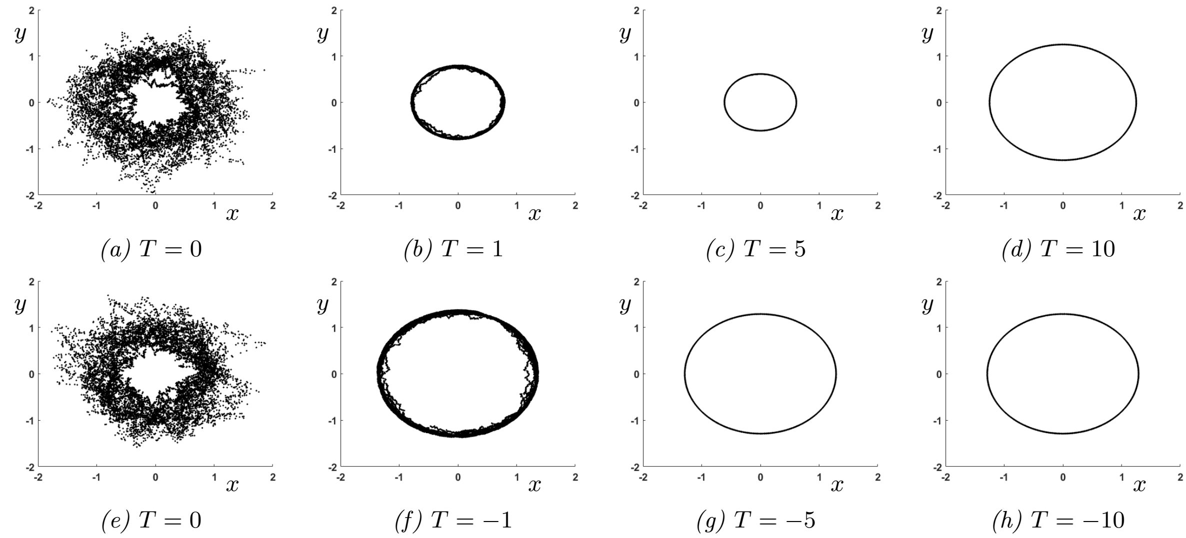


Figure 3.1: Numerical simulations in (x, y)-coordinates, using Euler-Marayama integration with step size $\mathrm{d}t = 10^{-2}$, of forward and pullback dynamics of system (3.1.4) for a set B of initial conditions generated by a trajectory of (3.1.4) ((a) and (e)). In (b)–(d), we show the numerical approximation of $\varphi(T, \omega, B)$ for some $\omega \in \Omega$, approaching the fiber $A(\theta_T\omega)$ of the random attractor, changing in forward time. In (f)–(h), we show the numerical approximation of $\varphi(-T, \theta_{-T}\omega, B)$ for some $\omega \in \Omega$, approaching the fiber $A(\omega)$ of the random attractor, fixed by the pullback mechanism.

Proof. Exercise. □

Additionally, we introduce the following notion:

Definition 3.1.8. A set $K \in \mathcal{D}$ is called an *absorbing set* for $D \in \mathcal{D}$, if there exists an *absorption time* $T_D(\omega) > 0$ such that for $\mathbb{P}$-almost all $\omega \in \Omega$

$$\varphi(t, \theta_{-t}\omega)D(\theta_{-t}\omega) \subset K(\omega) \quad \text{for all } t \geq T_D(\omega)\,.$$

We can formulate the following result concerning the properties of the set Ω_B for a set B which is absorbed by some compact set K:

Proposition 3.1.9. *Suppose $K, B \in \mathcal{D}$ are non-empty random sets with K absorbing B, where $K(\omega)$ is compact almost surely. Then for $\mathbb{P}$-almost all $\omega \in \Omega$ we have that*

(i) *$\Omega_B(\omega) \subset K(\omega)$ is non-empty and compact;*

(ii) *Ω_B is even strictly invariant, i.e. $\Omega_B(\theta_t\omega) \subset \varphi(t, \omega)\Omega_B(\omega)$ for all $t \geq 0$;*

(iii) *$\Omega_B(\omega)$ attracts B, i.e.*

$$\operatorname{dist}(\varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega), \Omega_B(\omega)) \to 0 \quad \text{as } t \to \infty.$$

Proof. Firstly, note that for any sequence $t_n \to \infty$ and sequence b_n with $b_n \in B(\theta_{-t_n}\omega)$, we have for $t_n \geq t_B(\omega)$ that $\varphi(t_n, \theta_{-t_n}\omega)b_n \in K(\omega)$. Hence, by compactness of $K(\omega)$, there is a convergent subsequence to some $y \in X$.

For (i), this means that the limit of such a sequence $y = \lim \varphi(t_n, \theta_{-t_n}\omega)b_n$ satisfies $y \in \Omega_B(\omega)$, such that $\Omega_B(\omega)$ is non-empty. Furthermore,

$$\Omega_B(\omega) \subset \bigcap_{\tau \geq T_B(\omega)} \overline{\bigcup_{t\geq\tau} \varphi(t, \theta_{-t}\omega)B(\theta_{-t}\omega)} \subset K(\omega).$$

(ii) Exercise.

(iii) If $\Omega_B(\omega)$ would not attract B, there would be $\delta > 0$, a sequence $t_n \to \infty$ and $b_n \in B(\theta_{-t_n}\omega)$ such that for all $n \in \mathbb{N}$

$$\mathrm{dist}(\varphi(t, \theta_{-t}\omega, b_n), \Omega_B(\omega)) \geq \delta.$$

But, as remarked above, $\varphi(t_n, \theta_{-t_n}\omega, b_n)$ has a convergent subsequence with a limit in $\Omega_B(\omega)$, which leads to a contradiction by continuity of $\varphi(t, \omega)$. □

We are ready to prove the main existence theorem for a random attractor, when taking $\mathcal{S} \subset \mathcal{D}$ to be the set of all bounded deterministic subsets of X. The theorem can be extended to all tempered random sets $\mathcal{D}$, but under heavy notation that we avoid here (see [9] for remarks on that):

Theorem 3.1.10 (Existence of random attractors)**.** *Suppose that (θ, φ) is a continuous random dynamical system on a Polish space X and that there exists a compact random set $K \in \mathcal{D}$ absorbing every bounded nonrandom set $B \subset X$. Then there exists a random pullback attractor A, given by*

$$A(\omega) = \overline{\bigcup_{B\subset X} \Omega_B(\omega)} \quad \textit{for almost all } \omega \in \Omega.$$

Furthermore, $\omega \mapsto A(\omega)$ is measurable with respect to $\mathcal{F}^0_{-\infty}$, i.e. the past of the system.

[**End of Lecture IX, 08.06.**]

Proof. Since $\Omega_B \subset K(\omega)$ by Proposition 3.1.9(i), for any bounded set B, we obtain that $A \in \mathcal{D}$ is a compact random set. Since

$$\omega \mapsto \bigcup_{B\subset X} \Omega_B(\omega)$$

is strictly invariant by Proposition 3.1.9(ii), we obtain that A is invariant by the continuity of φ, and, in fact, strictly invariant using compactness of A.

For measurability, note that for any $x \in X$, $t \geq 0$ and any $B \subset X$, the map

$$\omega \mapsto \mathrm{dist}(x, \varphi(t, \theta_{-t}\omega)B) = \inf_{y\in B} d(x, \varphi(t, \theta_{-t}\omega, y))$$

is measurable with respect to $\mathcal{F}^0_{-\infty}$ by separability of X and continuity of φ. For each $\tau \geq 0$

$$\mathrm{dist}\,(x, \cup_{t\geq\tau}\varphi(t, \theta_{-t}\omega)B) = \inf_{t\geq\tau} \mathrm{dist}(x, \varphi(t, \theta_{-t}\omega)B).$$

Hence, in discrete time, measurability of Ω_B with respect to $\mathcal{F}^0_{-\infty}$ follows immediately. For continuous time, one has to work with the projection theorem [7, Theorem III.23] to deduce that Ω_B is measurable. Since A can be obtained using only a countable number of Bs, the assertion is proved. □

This leaves open uniqueness. In fact, even without existence of an absorbing set, we can show that weak attractors (and thereby also random pullback attractors) are unique, if they exist.

For the proof, we need the following proposition:

Lemma 3.1.11 (Tightness of random compact sets)**.** *Suppose that $\omega \mapsto K(\omega)$ is a compact random set. Then for every $\varepsilon > 0$, there exists a (non-random) compact set $K_\varepsilon \subset X$ such that*

$$\mathbb{P}\{\omega : K(\omega) \subset K_\varepsilon\} \geq 1 - \varepsilon.$$

Proof. See Exercise 1 on QS 9. □

We now easily obtain the following uniqueness result.

Proposition 3.1.12. *Weak attractors (and hence pullback attractors) with respect to deterministic bounded sets (or also more generally with respect to tempered random sets $\mathcal{D}$) are unique in the sense, that if an RDS has two weak attractors, then they agree almost surely.*

Proof. Let $A, \tilde{A}$ be two weak random attractors. Since $\tilde{A}$ is a random compact set, by Lemma 3.1.11, for each $\varepsilon > 0$, there is a compact deterministic set K_ε such that

$$\mathbb{P}\{\tilde{A} \subset K_\varepsilon\} \geq 1 - \varepsilon.$$

Since A weakly attracts bounded, and in particular compact, sets, for all $\delta, \varepsilon > 0$, there is a $t_0(\delta, \varepsilon)$ such that for all $t \geq t_0$

$$\mathbb{P}\{\text{dist}(\varphi(t, \omega, K_\varepsilon), A(\theta_t\omega)) > \delta\} \leq \varepsilon.$$

Hence, we get that for all $t \geq t_0$

$$\mathbb{P}\{\text{dist}(\varphi(t, \omega, \tilde{A}(\omega)), A(\theta_t\omega)) > \delta\} \leq 2\varepsilon.$$

Using invariance of $\tilde{A}$ and θ_t, we obtain for all $t \geq t_0$

$$\mathbb{P}\{\text{dist}(\tilde{A}(\omega), A(\omega)) > \delta\} = \mathbb{P}\{\text{dist}(\tilde{A}(\theta_t\omega), A(\theta_t\omega)) > \delta\} \leq 2\varepsilon.$$

Since ε was arbitrary, we conclude

$$\mathbb{P}\{\text{dist}(\tilde{A}(\omega), A(\omega)) > \delta\} = 0 \quad \text{for all } \delta > 0,$$

which implies the claim. □

This holds not true for point attractors as we explain with the following illustrative example, which also is an example for distinguishing weak and pullback attractors.

Example 3.1.13. *Let $X := S^1$ be the unit circle which we identify with the interval $[0, 2\pi)$ equipped with the usual metric $d(x, y) := \min\{|x-y|, 2\pi - |x-y|\}$. Consider the SDE*

$$\mathrm{d}X_t = \cos(X_t)\,\mathrm{d}W_t^1 + \sin(X_t)\,\mathrm{d}W_t^2$$

on X, where W^1 and W^2 are independent standard Brownian motions. Then there exists a stable point $\omega \to S(\omega)$, which is measurable with respect to $\mathcal{F}_{-\infty}^0$, supporting a random invariant measure, and whose Lyapunov exponent is negative (see [4]). The random set $\{S(\omega)\}$ is a (minimal) weak point attractor of the RDS generated by the SDE. As the system is invertible, we can reverse time and use the same argument to find an unstable point $\omega \mapsto U(\omega)$, measurable with respect to $\mathcal{F}_0^\infty$ which is a weak point repeller. The basin of attraction of $\{S(\omega)\}$ is $X \setminus \{U(\omega)\}$ and that of $\{U(\omega)\}$ for the time-reversed flow is $X \setminus \{S(\omega)\}$. One can directly see that the weak attractor for all bounded deterministic sets (and all tempered random sets, more generally) is the full circle X.

In fact, also the unique pullback point attractor of φ is the whole space X, since $\Omega_x(\omega) = X$ almost surely for each fixed $x \in X$; this can be seen by noting that for each fixed $y \in X$, the process $t \mapsto \varphi(-t, \omega, y)$ is a Brownian motion on X and therefore hits x for some arbitrarily large values of t, showing that $y \in \Omega_x(\omega)$ for almost all $\omega \in \Omega$.

We mention one more important property of random attractors in connected spaces such as Euclidean space $\mathbb{R}^d$:

Proposition 3.1.14. *Suppose φ is an RDS on a connected space X. If φ has an attractor A for all bounded sets $B \subset X$, then $\mathbb{P}$-a.s. A is connected.*

Proof. Question sheet 9, Exercise 2. □

3.2 Attractors and invariant measures

Recall the set $C_\Omega(X)$ of functions $f : X \times \Omega \to \mathbb{R}$ such that $f(x, \cdot)$ is measurable for each $x \in X$, $f(\cdot, \omega)$ is continuous and bounded for each $\omega \in \Omega$ and $\omega \mapsto \sup\{|f(x, \omega)| : x \in X\}$ is integrable with respect to $\mathbb{P}$, where two such functions f and g are identified if $\mathbb{P}\{\omega : f(\cdot, \omega) \neq g(\cdot, \omega)\} = 0$ (measurable by continuity of f and g together with separability of X).

We define the *narrow toplogy* on the space of probability measures $\mathcal{P}_\Omega(X)$ with projection $\mathbb{P}$ on Ω to be the topology generated by the functions

$$\mu \mapsto \int_{X \times \Omega} f(x, \omega)\,\mathrm{d}\mu(x, \omega) = \mu(f)$$

for $f \in C_\Omega(X)$. Recall that the skew-product flow $(\Theta_t)_{t \in \mathbb{T}}$ acts as a flow of continuous transformations on $\mathcal{P}_\Omega(X)$.

Definition 3.2.1. Let π_X denote the canonical projection from $X \times \Omega$ onto X, i.e. for all $B \in \mathcal{B}(X)$,

$$(\pi_X \mu)(B) = \mu(B \times \Omega) = \int_\Omega \mu_\omega(B)\, \mathrm{d}\mathbb{P}(\omega) = \mathbb{E}[\mu_\omega(B)].$$

A subset $\Gamma \subset \mathcal{P}_\Omega(X)$ is said to be *tight* if $\pi_X \Gamma \subset \mathcal{P}(X)$ is tight, i.e. for every $\varepsilon > 0$ there exists a compact set $K_\varepsilon \subset X$ such that $\mu(K_\varepsilon \times \Omega) \geq 1 - \varepsilon$ for all $\mu \in \Gamma$.

We can now formulate Prohorov's Theorem for random measures:

Theorem 3.2.2. *Suppose that $\Gamma \subset \mathcal{P}_\Omega(X)$. Then Γ is tight if and only if it is relatively compact with respect to the narrow topology. In this case, it is also relatively sequentially compact.*

Proof. See for example [10, Chapter 4]. □

To show that such a set Γ is tight, we can use the following characterizations.

Proposition 3.2.3. *For $\Gamma \subset \mathcal{P}_\Omega(X)$, consider the following assertion:*

(a) For every $\varepsilon > 0$ there is a compact random set $\omega \mapsto K_\varepsilon(\omega)$ such that for every $\gamma \in \Gamma$

$$\gamma_\omega(K_\varepsilon(\omega)) \geq 1 - \varepsilon$$

for almost all $\omega \in \Omega$.

(b) For every $\varepsilon > 0$ there is a compact random set $\omega \mapsto K_\varepsilon(\omega)$ such that for every $\gamma \in \Gamma$

$$\int_\Omega \gamma_\omega(K_\varepsilon(\omega))\mathrm{d}\mathbb{P}(\omega) \geq 1 - \varepsilon.$$

(c) Γ is tight.

Then (a) implies (b), and (b) and (c) are equivalent.

Proof. Clearly (a) implies (b) and (c) implies (b) by taking $K_\varepsilon := K_\varepsilon(\omega)$ to be the set in Definition 3.2.1.

Assume (b) holds and, for $\varepsilon > 0$, choose $C_\varepsilon \subset X$ compact such that $\mathbb{P}\{K_{\varepsilon/2}(\omega) \subset C_\varepsilon\} \geq 1 - \varepsilon/2$, as in Lemma 3.1.11. Then for every $\gamma \in \Gamma$

$$\begin{aligned}
\pi_X \gamma(C_\varepsilon) &= \int_\Omega \gamma_\omega(C_\varepsilon)\, \mathrm{d}\mathbb{P}(\omega) \\
&\geq \int_{\{K_{\varepsilon/2}(\omega) \subset C_\varepsilon\}} \gamma_\omega(C_\varepsilon)\, \mathrm{d}\mathbb{P}(\omega) \\
&\geq\geq \int_{\{K_{\varepsilon/2}(\omega) \subset C_\varepsilon\}} \gamma_\omega(K_{\varepsilon/2}(\omega))\, \mathrm{d}\mathbb{P}(\omega) \\
&\geq (1 - \varepsilon/2)^2 \geq 1 - \varepsilon,
\end{aligned}$$

which shows the claim. □

We have now everything together to show the following theorem:

Theorem 3.2.4 (Existence of invariant measures supported on attractor)**.** *For an RDS (θ, φ) on a Polish space X with φ-forward invariant random compact set $\omega \mapsto A(\omega)$, i.e. $\varphi(t,\omega)A(\omega) \subset A(\theta_t\omega)$ for all $t \geq 0$, we have the following:*

(a) There exists an invariant measure μ which is supported on A, i.e. $\mu_\omega(A(\omega)) = 1$ almost surely.

(b) If $\omega \mapsto A(\omega)$ is measurable with respect to the past $\mathcal{F}^0_{-\infty}$, then there exists an invariant measure μ supported by A such that $\omega \mapsto \mu_\omega$ is measurable with respect to the past $\mathcal{F}^0_{-\infty}$.

In particular, this holds for random attractors of any form.

Proof. For (a), note that

$$\Gamma = \{\mu \in \mathcal{P}_\Omega(X) \,:\, \mu_\omega(A(\omega)) = 1 \text{ a.s.}\}$$

is tight with Proposition 3.2.3. Γ is also closed, since for any sequence μ^n in Γ converging to some $\mu \in \mathcal{P}_\Omega(X)$ in the narrow topology we have

$$1 = \limsup_{n\to\infty} \mu^n(A) \leq \mu(A) = \int_\Omega \mu_\omega(A(\omega))\,\mathrm{d}\mathbb{P}(\omega),$$

and, hence, $\mu \in \Gamma$. Clearly, Γ is convex and invariant under $\Theta_t, t \geq 0$. Thus, the statement follows from the Markov-Kakutani fixed point theorem (see e.g. [12, Theorem V., 10.6.]).

For (b), observe that the set of all probability measures with $\omega \mapsto \mu_\omega$ measurable with respect to the past $\mathcal{F}^0_{-\infty}$ is a closed subset of $L^\infty(\Omega; \mathcal{P}(X))$. We have seen before that the set of $\mathcal{F}^0_{-\infty}$-measurable measures is invariant under the linear continuous action induced by Θ_t. Hence, it is enough to establish a $\mathcal{F}^0_{-\infty}$-measurable measure $\omega \mapsto \mu_\omega$ supported by $A(\omega)$; we obtain this by choosing a measurable selection $\omega \mapsto x(\omega) \in A(\omega)$ and setting $\mu_\omega = \delta_{x(\omega)}$. □

Using the Correspondence Theorem 1.5.6, we observe that for Markov RDS (past and future are independent, and correspondence with Markov process) with random pullback attractor, we can directly obtain an invariant probability measure ρ for the Markov semi-group $(P_t)_{t\geq 0}$ (stationary measure) via $\rho = \mathbb{E}[\mu_\omega]$. If there exists a unique invariant probability measure ρ for the Markov semi-group $(P_t)_{t\geq 0}$, then the invariant Markov measure, supported on A, is uniquely given by the Correspondence Theorem 1.5.6.

We say that a Markov semi-group is strongly mixing if

$$P_t f(x) \xrightarrow{t\to\infty} \int_X f(y)\rho(\mathrm{d}y) \quad \text{for all continuous and bounded } f : X \to \mathbb{R} \text{ and } x \in X\,.$$

Similarly, we say that the associated cocycle φ is strongly mixing if the law of $\varphi(t, \cdot, x)$ converges to ρ for $t \to \infty$ for all $x \in X$. More generally, we introduce

$$E_0 := \left\{ x \in X \,:\, \lim_{t\to\infty} \hat{P}_t(x, \cdot) = \rho \right\},$$

where $\hat{P}_t(x, \cdot)$ denotes the transition probability and convergence is to be understood in the weak* sense. We cite the following additional insight without a proof:

Proposition 3.2.5. *Assume that a Markov RDS has a unique invariant Markov measure μ such that $\tilde{A}(\omega) = \operatorname{supp} \mu_\omega$ is almost surely compact. Then $\tilde{A}$ is a weak point attractor of the set E_0. In particular, if φ is strongly mixing, then $\tilde{A}$ is a minimal weak random point attractor.*

Proof. See [14, Proposition 2.20]. □

If A is a random pullback attractor for tempered sets, then obviously $\tilde{A}(\omega) = \operatorname{supp} \mu_\omega \subset A(\omega)$ for almost all $\omega \in \Omega$, see also Example 3.1.13.
[**End of Lecture X, 15.06.**]

Concerning the support of μ_ω, we can additionally make the following observation:

Lemma 3.2.6. *Let μ_ω be the disintegration of an ergodic invariant Markov measure μ for an injective Markov RDS (θ, φ), then we have the following: either μ_ω consists of finitely many atoms of the same mass $\mathbb{P}$-a.s., i.e. there is an $N \in \mathbb{N}$ and $\mathcal{F}^0_{-\infty}$-measurable random variables $a_1, \ldots, a_N$ such that*

$$\mu_\omega = \frac{1}{N} \sum_{i=1}^{N} \delta_{a_i(\omega)},$$

or μ_ω does not have point masses almost surely.

Proof. Assume that μ_ω has point masses with positive probability and define $g(\omega, x) := \mu_\omega(\{x\})$. Then for μ-almost all (ω, x), we have

$$g(\Theta_t(\omega, x)) = \mu_{\theta_t\omega}(\{\varphi(t, \omega, x)\}) = \mu_\omega\left(\varphi(t, \omega, \cdot)^{-1}\{\varphi(t, \omega, x)\}\right) = g(\omega, x),$$

where in the last equality, we have used injectivity. Due to ergodicity, we can can conclude that g is constant μ-almost surely. Hence, all point masses of μ_ω have the same mass $m \in \mathbb{R}^+$, $\mathbb{P}$-almost surely. By assumption, we have that $m > 0$. Then we have for almost all $\omega \in \Omega$

$$m = \int_X g(\omega, x) \, \mathrm{d}\mu_\omega(x) = \int_X \mu_\omega(\{x\}) \, \mathrm{d}\mu_\omega(x) = N(\omega) m^2,$$

where $N(\omega)$ denotes the number of point masses of μ_ω. This implies $N(\omega) = 1/m$ almost surely, which finishes the proof. □

We can give a criterion for the Markov measures being discrete that can then be linked to negative top Lyapunov exponents. We will also make a similar statement about the case of positive top Lyapunov exponent being linked to the Markov measure having no point masses.

Definition 3.2.7. Let $U \subset X$ be a deterministic non-empty open set.

1. We say that φ is *asymptotically stable* on U, if there exists a sequence $t_n \to \infty$ such that

$$\mathbb{P}\left(\omega \in \Omega : \lim_{n \to \infty} \operatorname{diam}(\varphi(t_n, \omega, U)) = 0\right) > 0 \,.$$

2. More generally, we say that φ is *weakly asymptotically stable on* U if there exsist a (deterministic) sequence $t_n \to \infty$ and a set $\mathcal{M} \subset \Omega$ of positive $\mathbb{P}$-measure such that for all $x, y \in U$

$$1_{\mathcal{M}}(\cdot)d(\varphi(t_n, \cdot, x), \varphi(t_n, \cdot, y)) \to 0 \quad \text{as } n \to \infty. \tag{3.2.1}$$

in probability.

Proposition 3.2.8. *Let* (θ, φ) *be an injective Markov RDS on a Polish space* X *with unique stationary measure* ρ *and associated Markov measure* μ_ω.

(a) If φ *is weakly asymptotically stable on* U *with* $\rho(U) > 0$, *then* μ_ω *is discrete.*

(b) If in addition φ *is strongly mixing, then there is an* $N \in N$ *and* $\mathcal{F}^0_{-\infty}$*-measurable random variables* $a_1, \dots, a_n$ *such that*

$$A(\omega) = \operatorname{supp}(\mu_\omega) = \{a_i(\omega) : i = 1, \dots, N\}$$

is a minimal weak point attractor.

Proof. (a) With Lemma 3.2.6, we only have to show that μ_ω has point mass with positive probability. Let $\Delta := \{(x, x) : x \in X\} \subset X \times X$ be the diagonal in $X \times X$ and $\psi : \hat{X} := (X \times X) \setminus \Delta \to [0, \infty)$ be measurable such that $\psi(x, y) \to \infty$ for $d(x, y) \to 0$ and

$$\mathbb{E} \int_{\hat{X}} \psi(x, y) \, \mathrm{d}\mu_\omega(x) \mathrm{d}\mu_\omega(y) < \infty.$$

In order to make sure the existence of such a ψ, we define $\nu := \mathbb{E}[\mu_\omega \otimes \mu_\omega]$ on $X \times X$ and observe that $\nu(\Delta^\varepsilon \setminus \Delta) \to 0$ as $\varepsilon \to 0$, where Δ^ε denotes the ε-neighbourhood of Δ. Choosing $\varepsilon_k \to 0$ with $\varepsilon_k \leq \varepsilon_0 = 1$ such that $\nu(\Delta^\varepsilon \setminus \Delta) \leq e^{-k}$, we set

$$\psi(x, y) = \begin{cases} k, & \text{if } (x, y) \in \Delta^{\varepsilon_k} \setminus \Delta^{\varepsilon_{k+1}} \\ 0, & \text{if } |x - y| \geq 1. \end{cases}$$

Let U, $\mathcal{M}$ and t_n be as in the definition of weak asymptotic stability and define

$$C(n, x, y, R) := \{\omega \in \Omega : \psi(\varphi(t_n, \omega, x), \varphi(t_n, \omega, y)) \geq R\}$$

such that we observe for all $x, y \in U$

$$\liminf_{n \to \infty} \mathbb{P}\left(C(n, x, y, R)\right) \geq \mathbb{P}(\mathcal{M}).$$

Observe that

$$\begin{aligned}
&\mathbb{E}\int_{\hat{X}} \psi(x,y)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&\geq \mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)\psi(\varphi(t_n,\omega,x),\varphi(t_n,\omega,y))\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&\geq R\mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)1_{C(n,x,y,R)}(\omega)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y).
\end{aligned}$$

Since μ_ω is $\mathcal{F}_{-\infty}^0$-measurable, $C(n,x,y,R)$ is $\mathcal{F}_0^\infty$-measurable and $\mathcal{F}_{-\infty}^0$ and $\mathcal{F}_0^\infty$ are independent, we conclude

$$\begin{aligned}
&\mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)1_{C(n,x,y,R)}(\omega)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&= \mathbb{E}\mathbb{E}\left[\int_{\hat{X}} 1_U(x)1_U(y)1_{C(n,x,y,R)}(\omega)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)|\mathcal{F}_{-\infty}^0\right]\\
&= \mathbb{E}\tilde{\mathbb{E}}\int_{\hat{X}} 1_U(x)1_U(y)1_{C(n,x,y,R)}(\tilde{\omega})\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&= \mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)\mathbb{P}(C(n,x,y,R))\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)
\end{aligned}$$

Taking the lim inf and using Fatou's Lemma we obtain

$$\begin{aligned}
&\mathbb{E}\int_{\hat{X}} \psi(x,y)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&\geq R\mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)\liminf_{n\to\infty}\mathbb{P}(C(n,x,y,R))\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&\geq \mathbb{P}[\mathcal{M}]R\mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y).
\end{aligned}$$

If we now assume that μ_ω has no point masses, then $(\mu_\omega\otimes\mu_\omega)(\Delta)=0$ and therefore

$$\begin{aligned}
\mathbb{E}\int_{\hat{X}} 1_U(x)1_U(y)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y) &= \mathbb{E}\int_{X\times X} 1_U(x)1_U(y)\,\mathrm{d}\mu_\omega(x)\mathrm{d}\mu_\omega(y)\\
&= \mathbb{E}(\mu_\omega(U)^2)\geq\rho(U)^2>0.
\end{aligned}$$

Since $R>0$ is arbitrary, we obtain a contradiction, and, hence, the proof of (a) is complete, considering Lemma 3.2.6.

(b) This follows from (a) in combination with Proposition 3.2.5. □

Let us now assume we are in the setting of a smooth ergodic RDS induced by an SDE in $\mathbb{R}^d$ such that we can apply the Multiplicative Ergodic Theorem and the Stable Manifold Theorem. It is then easy to observe that $\lambda_1>0$ implies that φ is asymptoatically stable, in particular weakly asymptotically stable, (Exercise!!!) such that we find the minimal weak attractor to consist of points, being the equidistributed support of the invariant Markov measure.

Concerning general set attractors, we can state the following (weak) synchronization statement:

Theorem 3.2.9 (Collapse of the random attractor)**.** *We assume that a random dynamical system* (θ, φ) *is*

(i) asymptotically stable *on a fixed non-empty open set* $U \subset \mathbb{R}^d$*, in the sense that there exists a sequence* $t_n \to \infty$ *such that*

$$\mathbb{P}\left(\omega \in \Omega : \lim_{n\to\infty} \operatorname{diam}(\varphi(t_n, \omega, U)) = 0\right) > 0\,.$$

(ii) swift transitive*, i.e. for all* $x, y \in \mathbb{R}^d$ *and* $r > 0$*, there exists a* $t > 0$ *such that*

$$\mathbb{P}\big(\omega \in \Omega : \varphi(t, \omega, B_r(x)) \subset B_{2r}(y)\big) > 0\,.$$

(iii) contracting on large sets*, i.e. for all* $R > 0$*, there exist* $y \in \mathbb{R}^d$ *and* $t > 0$ *such that*

$$\mathbb{P}\left(\omega \in \Omega : \operatorname{diam}(\varphi(t, \omega, B_R(y))) \le \tfrac{R}{4}\right) > 0\,.$$

Suppose further that (θ, φ) *has a weak random attractor* A *with* $\mathcal{F}_{-\infty}^0$*-measurable fibers. Then* $A(\omega)$ *is a singleton* $\mathbb{P}$*-almost surely.*

Proof. See [14, Theorem 2.14]. □

3.3 Some properties of chaotic random attractors

In the following, we are concerned with random attractors for differentiable RDS on $X = \mathbb{R}^d$ (one could also consider smooth manifolds), where the first Lyapunov exponent $\lambda_1 > 0$. We will call such attractors *chaotic attractors*. A first justification for this term is given by the following observation. The proof uses similar arguments as for the (weakly) asymptotically stable case of $\lambda_1 < 0$, also considering the two-point motion on $X \times X$ and the diagonal Δ.

Theorem 3.3.1 (Positive λ_1 implies atomless μ_ω)**.** *Consider a smooth ergodic RDS induced by an SDE in* $X = \mathbb{R}^d$ *(or on a compact manifold* $X = M$*) such that we can apply the Multiplicative Ergodic Theorem, and assume that the top Lyapunov exponent* λ_1 *is positive. Then for* $\nu = \mathbb{E}[\mu_\omega \otimes \mu_\omega]$ *as above, we have*

$$\nu\left(\hat{X}\right) = 1$$

or equivalently μ_ω *is atomless almost surely.*

Proof. See [13, Section 5.1]. □

3.3.1 Pesin's formula

In this section, we want to investigate what positive Lyapunov exponents imply for the entropy of the system. We will follow [6, 19, 24].

Entropy for discrete time systems

Firstly, we formulate the statements for $\mathbb{R}^d$ and random dynamical systems in discrete time generated by composed maps $\left\{f_\omega^n : n \geq 0, \omega \in \left(\Omega^{\mathbb{N}}, \mathcal{B}(\Omega)^{\mathbb{N}}, \nu^{\mathbb{N}}\right)\right\}$ which will be referred to as $\mathcal{X}^+(\mathbb{R}^d, \nu)$ (we will also consider two-sided time in a moment). Here, Ω denotes the set of two-times differentiable diffeomorphisms on $\mathbb{R}^d$ with the topology induced by uniform convergence on compact sets for all derivatives up to order 2. The maps are i.i.d. with law ν, and for a sequence $\omega = (f_0(\omega), f_1(\omega), \dots) \in \Omega^{\mathbb{N}}$ the compositions are given as

$$f_\omega^0 = \mathrm{id}\,, \quad f_\omega^n = f_{n-1}(\omega) \circ f_{n-2}(\omega) \circ \cdots \circ f_0(\omega)\,.$$

Recall from the exercises:

Definition 3.3.2 (Stationary measure)**.** A Borel probability measure ρ on $\mathbb{R}^d$ is called a stationary measure of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ if

$$\rho(\cdot) = \int_\Omega \rho\left(f^{-1}(\cdot)\right) \nu(\mathrm{d}f)\,.$$

Note in particular that the associated Markov chain X_n has ρ as its stationary measure in this case.

If ξ is a finite partition of a Lebesgue space $(X, \mathcal{B}, \mu)$ and $C_1, \dots, C_k$ denote the elements of ξ, we define the *entropy* of ξ with respect to μ by

$$H_\mu(\xi) = -\sum_{j=1}^k \mu(C_j) \log(\mu(C_j))\,.$$

Furthermore, for two partitions ξ_1 and ξ_2 we define

$$\xi_1 \vee \xi_2 = \{A \cap B \,:\, A \in \xi_1, B \in \xi_2\}\,,$$

such that elements of $\bigvee_{i=0}^{n-1} (f_\omega^i)^{-1}\xi$ are of the form

$$\{x \,:\, x \in C_{j_0}, f_\omega x \in C_{j_1}, \dots, f_\omega^{n-1} x \in C_{j_{n-1}}\}$$

for some $(j_0, \dots, j_{n-1})$ sometimes called the address of the orbit.

Following [6, 24], we define the entropy of a random dynamical system in the following way:

Definition and Lemma 3.3.3 (Entropy)**.** For any finite partition ξ of $\mathbb{R}^d$ and stationary measure ρ of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ the limit

$$h_\rho\left(\mathcal{X}^+(\mathbb{R}^d, \nu), \xi\right) := \lim_{n\to\infty} \frac{1}{n} \int_{\Omega^{\mathbb{N}}} H_\rho\left(\bigvee_{k=0}^{n-1} (f_\omega^k)^{-1}\xi\right) \nu^{\mathbb{N}}(\mathrm{d}\omega)$$

exists. The number $h_\rho\left(\mathcal{X}^+(\mathbb{R}^d, \nu), \xi\right)$ is called the entropy of $\mathcal{X}^+(\mathbb{R}^d, \nu)$ with respect to ξ. The number

$$h_\rho\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right) := \sup_\xi h_\rho\left(\mathcal{X}^+(\mathbb{R}^d,\nu),\xi\right)$$

is called the entropy of $\mathcal{X}^+(\mathbb{R}^d,\nu)$.

Consider the product spaces $\Omega^{\mathbb{N}} \times \mathbb{R}^d$ and $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ with the respective product σ-algebras, as usual.

[**End of Lecture XI, 22.06.**]

As before, we denote the left shift operator on $\Omega^{\mathbb{N}}$ and $\Omega^{\mathbb{Z}}$ by θ, i.e.

$$f_n(\theta\omega) = f_{n+1}(\omega)$$

for all $\omega \in \Omega^{\mathbb{N}}$, $n \in \mathbb{N}$ and $\omega \in \Omega^{\mathbb{Z}}$, $n \in \mathbb{Z}$ respectively, and the associated skew product systems on $\Omega^{\mathbb{N}} \times \mathbb{R}^d$ or $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ respectively by

$$\Theta(\omega,x) = (\theta\omega, f_0(\omega)x).$$

Recall from before that ρ is a stationary measure for $\mathcal{X}^+(\mathbb{R}^d,\nu)$ iff $\nu^{\mathbb{N}} \times \rho$ is an invariant measure for the one-sided skew product system Θ on $\Omega^{\mathbb{N}} \times \mathbb{R}^d$. Furthermore, by the correspondence theorem 1.5.6, we again get the existence of a unique Borel probability measure μ on $\Omega^{\mathbb{Z}} \times \mathbb{R}^d$ such that $\Theta\mu = \mu$ and $P\mu = \nu^{\mathbb{N}} \times \rho$, where P denotes the projection to the measures on $\Omega^{\mathbb{N}} \times \mathbb{R}^d$.

As we have seen many times before, we can consider the associated Markov process and, hence, when also considering two-sided time, we can separate past and future σ-algebras. For reasons of brevity, we will not do this here in more detail, but we want to remark that one may also consider the conditional entropy with respect to the future of the two-sided time system and show that this is the same as $h_\mu\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right)$.

Furthermore, we recall Oseledets' Multiplicative Ergodic Theorem 2.5.3 and apply it to (Θ,μ) under assuming the integrability condition, without μ being necessarily ergodic. There is an Oseledets splitting

$$\mathbb{R}^d = E_1(\omega,x) \oplus \cdots \oplus E_{p(\omega,x)}(\omega,x)$$

such that for μ-a.e. (ω,x)

$$\lim_{n\to\pm\infty} \frac{1}{n} \ln \|\mathrm{D}f_\omega^n v\| = \lambda_i(\omega,x) \quad \text{if } 0 \neq v \in E_i(\omega,x)\,. \tag{3.3.1}$$

The maps $(\omega,x) \mapsto p(\omega,x), \lambda_i(\omega,x), \dim E_i(\omega,x)$ are measurable and constant along orbits of Θ. In fact, there are functions $p, \lambda_i, m_i : \mathbb{R}^d \to \mathbb{R}$ such that for μ-a.e. (ω,x)

$$p(\omega,x) = p(x),\;\; \lambda_i(\omega,x) = \lambda_i(x) \text{ and } \dim E_i(\omega,x) = m_i(x)\,,$$

where d_i is the multiplicity of λ_i. This can be seen by using ergodic decompositions of invariant measures; we refer to Kifer [19, Appendix A.1] for the ergodic decomposition and to [24, Chapter 1, Remark 3.1] for its application. As before, if μ is ergodic, these functions are constant, i.e. the

x-dependence also vanishes.

In [6] we find five integrability assumptions that amount to the fact that all derivatives are satisfying the conditions of the MET in forward and backward time. We call this assumption (A). It is not necessary for our purposes to list these here specifically as the assumptions are satisfied for sufficiently regular stochastic flows. The theorem confirming Pesin's formula in this setting reads as follows:

Theorem 3.3.4 (Pesin's formula)**.** *Let $\mathcal{X}^+(\mathbb{R}^d,\nu)$ be a random dynamical system which has an absolutely continuous stationary probability measure ρ and satisfies (A). Then we have*

$$h_\rho\left(\mathcal{X}^+(\mathbb{R}^d,\nu)\right) = \int_{\mathbb{R}^d}\sum_i \lambda_i(x)^+ m_i(x)\rho(\mathrm{d}x)\,, \tag{3.3.2}$$

where $\lambda_i(x)^+$ are the positive Lyapunov exponents and $m_i(x)$ their multiplicities.

Proof. See [6]. □

Entropy for stochastic flows

Assume now that we are in the situation of Theorem 1.4.5 for an SDE (1.4.7) inducing a C^k random dynamical system (θ,φ). We relate the system to a stochastic flow of C^k diffeomorphisms in the sense of [22] by defining the maps

$$\tilde\varphi : \mathbb{R}_0^+\times\mathbb{R}_0^+\times\bar\Omega\times\mathbb{R}^d\to\mathbb{R}^d, \quad \tilde\varphi_{s,t}(\bar\omega,x) = \varphi(t-s,\theta_s\bar\omega,x)\,,$$

where $(\bar\Omega,\mathcal{F},\mathbb{P})$ is the canonical Wiener space. By this definition $\tilde\varphi_{s,t}(\bar\omega,\cdot)$ is a C^k diffeomorphism for each $s,t\geq 0$ and $\bar\omega\in\bar\Omega$.

Now let $k\geq 2$ and define Ω as above as the space of C^2 diffeomorphisms equipped with the uniform topology on compact sets. In this case, the measure

$$\nu(\cdot) = \mathbb{P}\{\bar\omega\in\bar\Omega\,:\,\tilde\varphi_{0,1}(\bar\omega,\cdot)\in\cdot\} \tag{3.3.3}$$

on $(\Omega,\mathcal{B}(\Omega))$ and the random diffeomorphisms

$$f_0(\omega) = \tilde\varphi_{0,1}(\bar\omega,\cdot) = \varphi(1,\bar\omega,\cdot) \tag{3.3.4}$$

generate, as before, a random dynamical system in discrete time

$$\mathcal{X}^+(\mathbb{R}^d,\nu) = \left\{f_\omega^n\,:\,n\geq 0, \omega\in\left(\Omega^{\mathbb{N}},\mathcal{B}(\Omega)^{\mathbb{N}},\nu^{\mathbb{N}}\right)\right\}\,.$$

Observe that the measure ρ is stationary for this system if for any set $A\in\mathcal{B}(\mathbb{R}^d)$

$$\rho(A) = \int_{\bar\Omega}\rho\left((\tilde\varphi_{0,1}(\bar\omega,\cdot))^{-1}(A)\right)\,\mathbb{P}(\mathrm{d}\bar\omega)\,.$$

Let $P(t, x, \cdot)$ denote the transition probabilities associated to the stochastic differential equation. Then we make the following observation:

Lemma 3.3.5. *Any invariant probability measure ρ for the Markov semi-group associated to the stochastic differential equation* (1.4.7) *is stationary for the induced discrete time system $\mathcal{X}^+(\mathbb{R}^d, \nu)$.*

Proof. For all $A \in \mathcal{B}(\mathbb{R}^d)$ we have with Fubini that

$$\begin{aligned} \rho(A) &= \int_{\mathbb{R}^d} P(1, x, A)\, \rho(\mathrm{d}x) = \int_{\mathbb{R}^d} \int_{\bar{\Omega}} \mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega}, x))\, \mathbb{P}(\mathrm{d}\bar{\omega})\rho(\mathrm{d}x) \\ &= \int_{\bar{\Omega}} \int_{\mathbb{R}^d} \mathbb{1}_A(\tilde{\varphi}_{0,1}(\bar{\omega}, x))\, \rho(\mathrm{d}x)\mathbb{P}(\mathrm{d}\bar{\omega}) = \int_{\bar{\Omega}} \rho\left((\tilde{\varphi}_{0,1}(\bar{\omega}, \cdot))^{-1}(A)\right)\, \mathbb{P}(\mathrm{d}\bar{\omega})\,, \end{aligned}$$

which shows the claim. □

Using the relations between stochastic flows, continuous-time random dynamical systems and discrete time random dynamical systems as explained above, we can now formulate Pesin's formula for random dynamical systems induced by stochastic differential equations:

Theorem 3.3.6 (Pesin's formula for SDEs)**.** *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation of the form* (1.4.7) *and let the assumptions of Theorem 1.4.5 be satisfied. Let further be ρ an absolutely continuous stationary probability measure satisfying*

$$\int_{\mathbb{R}^d} (\log(|x| + 1)^{1/2}\, \rho(\mathrm{d}x) < \infty\,. \tag{3.3.5}$$

Then the discrete time random dynamical system $\mathcal{X}^+(\mathbb{R}^d, \nu)$ associated with (θ, φ) satisfies (A) and, hence,

$$h_\rho\left(\mathcal{X}^+(\mathbb{R}^d, \nu)\right) = \int_{\mathbb{R}^d} \sum_i \lambda_i(x)^+ m_i(x) \rho(\mathrm{d}x)$$

holds.

Proof. A direct consequence of [6, Theorem 9.1]. □

In particular, we obtain that RDS with random attractors that support invariant Markov measures and are associated with positive Lyapunov exponents, can also be associated with positive entropy.

3.3.2 SRB measures

Another more profound notion of chaos could be given by showing the SRB-property of the random measures μ_ω, which are the disintegrations of an invariant probability measure μ for the two-sided skew product system, i.e. $\mu(\mathrm{d}x, \mathrm{d}\omega) = \mu_\omega(\mathrm{d}x)\nu^{\mathbb{Z}}(\mathrm{d}\omega)$.

Let us assume we are in exactly the same setting of a discrete time random dynamical system as above with the only difference that the state space is a compact smooth manifold M, calling such a system $\mathcal{X}^+(M, \nu)$, generated by C^2 diffeomorphisms and a law ν. Let the sample measures μ_ω be associated with a stationary measure ρ and write $E_i(\omega, x)$ for the Oseledets spaces corresponding with the Lyapunov exponents $\lambda_i(x)$. We follow [23] for the following definitions and results.

Definition 3.3.7. Let $(\omega, x) \in \Omega^{\mathbb{Z}} \times M$ be s.t. $\lambda_i(x) > 0$ for some i. Then the unstable manifold and the stable manifold of the skew product flow Θ at (ω, x) are given by

$$W^u(\omega, x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{-n} x, f_\omega^{-n} y) < 0 \right\},$$
$$W^s(\omega, x) = \left\{ y \in M \,:\, \limsup_{n\to\infty} \frac{1}{n} \ln d(f_\omega^{n} x, f_\omega^{n} y) < 0 \right\}.$$

At μ-a.e. (ω, x) with $\lambda_i(x) > 0$ for some i, $W^u(\omega, x)$ is a $\left(\sum_{\lambda_i > 0} \dim E_i(\omega, x)\right)$-dimensional C^2 immersed submanifold of M; this follows from the time-reversed version of Theorem 2.6.1. We set $W^u(\omega, x) = \{x\}$ if $\lambda_i(x) \leq 0$ for all i. If η is a partition of $\Omega^{\mathbb{Z}} \times M$, η_ω denotes the restriction of η to the fibre $\{\omega\} \times M$ which is a partition of M. We write $\eta_\omega(x)$ for the element of η_ω that contains x.

Definition 3.3.8. A measurable partition η of $\Omega^{\mathbb{Z}} \times M$ is called subordinate to W^u if for μ-a.e. (ω, x), $\eta_\omega(x) \subset W^u(\omega, x)$ and contains an open neighbourhood of x in $W^u(\omega, x)$, this neighbourhood being taken in the submanifold topology of $W^u(\omega, x)$.

Identifying σ-algebras with their generating partitions, we denote by σ the partition of $\Omega^{\mathbb{Z}} \times M$ into sets of the form $\{\omega\} \times M$. If η is a partition subordinate to W^u, μ disintegrates into a system of conditional measures on elements of $\eta \vee \sigma$, denoted by $\left\{\mu^{\eta\vee\sigma}_{(\omega,x)}\right\}$. For μ-a.e. (ω, x) we have the identification $\mu^{\eta\vee\sigma}_{(\omega,x)} = (\mu_\omega)^{\eta_\omega}_x$. Finally let $\lambda_{W^u(\omega,x)}$ denote the Riemannian measure on $W^u(\omega, x)$.

Definition 3.3.9 (SRB measures)**.** The sample measures μ_ω are called SRB measures or absolutely continuous conditional measures on W^u-manifolds if for every measurable partition η subordinate to W^u, $\mu^{\eta\vee\sigma}_{(\omega,x)}$ is absolutely continuous with respect to $\lambda_{W^u(\omega,x)}$ for μ-a.e. (ω, x).

Ledrappier & Young [23] can then prove the following statement.

Theorem 3.3.10. *Suppose the stationary measure ρ of the random dynamical system $\mathcal{X}^+(M, \nu)$ is absolutely continuous with respect to the Lebesgue measure and $\int \lambda_1 \mathrm{d}\rho > 0$. Then the sample measures μ_ω are SRB measures.*

Similarly to before, we can then formulate the following corollary for stochastic differential equations. As usual for the manifold case, we use the Stratonovich integral due to its classical properties in terms of the chain rule:

Corollary 3.3.11. *Let (θ, φ) be a random dynamical system induced by a stochastic differential equation with C^2 coefficients and stationary absolutely continuous probability distribution ρ on a compact manifold M. Let further $\int \lambda_1 \mathrm{d}\rho > 0$ and $\tilde{\mu}$ be the invariant probability measure of (θ, φ) corresponding to ρ. Then the sample measures $\tilde{\mu}_{\bar{\omega}}$ are SRB measures.*

Proof. By Lemma 3.3.5, ρ is stationary for the induced discrete time system $\mathcal{X}^+(M,\nu)$. Then the claim follows immediately from Theorem 3.3.10 if we can show that $\tilde{\mu}_{\bar{\omega}}$ are the disintegrations μ_ω of the invariant measure μ of $\mathcal{X}^+(M,\nu)$ associated to ρ. By the correspondence theorem, the probability measures μ_ω are given by

$$\mu_\omega = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

However, identifying ω and $\bar{\omega}$ via relation (3.3.4) we also have that $\tilde{\mu}_{\bar{\omega}}$ satisfies

$$\tilde{\mu}_{\bar{\omega}} = \lim_{n\to\infty} \varphi(n, \theta_{-n}\bar{\omega}, \cdot)\rho = \lim_{n\to\infty} f^n_{\tau^{-n}\omega}\rho \text{ for } \nu^{\mathbb{Z}}\text{-a.e. } \omega\,.$$

Hence, the claim follows. □

Chapter 4

Bifurcations in random dynamical systems

4.1 Bifurcations via topological equivalence and D-bifurcations

The classical definition in deterministic dynamical systems for finding a bifurcation at some parameter value $\alpha_0 \in \mathbb{R}$, is that the respective system is topologically not equivalent for $\alpha \leq \alpha_0$ and $\alpha > \alpha_0$ in some neighbourhood of α_0, i.e. there is no conjugacy between the two parameter regimes. We translate the notion of topological equivalence into the random context as follows:

Definition 4.1.1 (Topological equivalence)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and $(X_1, d_1), (X_2, d_2)$ be Polish spaces. Then, for $\tilde{\mathbb{T}} \in \{\mathbb{T}, \mathbb{T}_0^+\}$, the random dynamical systems $(\theta, \varphi_1 : \tilde{\mathbb{T}} \times \Omega \times X_1 \to X_1)$ and $(\theta, \varphi_2 : \tilde{\mathbb{T}} \times \Omega \times X_2 \to X_2)$ are called *topologically equivalent* if there exists a conjugacy $h : \Omega \times X_1 \to X_2$ satisfying the following properties:

(i) For almost all $\omega \in \Omega$, the function $x \mapsto h(\omega, x)$ is a homeomorphism from $X_1 \to X_2$.

(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.

(iii) The random dynamical systems φ_1 and φ_2 are cohomologous, i.e.

$$\varphi_2(t, \omega, h(\omega, x)) = h(\theta_t \omega, \varphi_1(t, \omega, x)) \text{ for all } t \in \tilde{\mathbb{T}}, x \in X_1 \text{ and almost all } \omega \in \Omega.$$

This allows us to introduce bifurcations for RDS, already for multidimensional bifurcation parameters $\alpha \in \mathbb{R}^k$. We will focus on state space $\mathbb{R}^d$ in the following:

Definition 4.1.2 (Bifurcation point)**.** Let $(\varphi_\alpha)_{\alpha \in \mathbb{R}^k}$ be a family of continuous RDS on $\mathbb{R}^d$. A parameter value α_0 is called a *bifurcation point* of the family if the family is not structurally stable at α_0, i.e. if in any neighbourhood of α_0 there are parameter values α such that φ_α and φ_{α_0} are not topologically equivalent.

We can easily make the following observation:

Lemma 4.1.3. *Let φ_1 and φ_2 be topologically equivalent as in Definition 4.1.1. Then if φ_1 has an invariant measure μ with disintegrations μ_ω, then φ_2 has an invariant measure ν with disintegrations ν_ω given by*

$$\nu_\omega(B) = \mu_\omega\left(h(\omega,\cdot)^{-1}B\right)$$

for all $B \in \mathcal{B}(X_2)$ (and, of course, vice versa).

Proof. Exercise on Question Sheet 11. □

Remark 4.1.4. This one-to-one correspondence is very helpful for us to determine bifurcations, given all knowledge about invariant measures that we have collected in the previous chapters. In particular, considering results of the previous chapter, we have the following: If we consider a differentiable ergodic RDS, say for example from a strongly mixing SDE, that depends on a parameter α, and where the first Lyapunov exponent $\lambda_1(\alpha)$ crosses through zero at some α_0, then the invariant measure changes from a discrete to a diffuse support $A(\omega) = \operatorname{supp}\mu_\omega$ (which is a random point attractor); in particular, the RDS exhibits a bifurcation at α_0.

In Arnold's book [2] you find the following more specific definition that has been motivated by SDEs with multiplicative noise:

Definition 4.1.5 (D-bifurcation point)**.** Let $(\varphi_\alpha)_{\alpha\in\mathbb{R}^k}$ be a family of (local) C^1 RDS in $\mathbb{R}^d$ with a respective family of ergodic invariant measures μ_α. A parameter value α_D is calles a *D-bifurcation point* of $(\varphi_\alpha, \mu_\alpha)_{\alpha\in\mathbb{R}^k}$ if in each neighbourhood of α_D there is an α for which there is an invariant measure $\nu_\alpha \neq \mu_\alpha$ with $\nu_\alpha \to \mu_{\alpha_D}$ weakly as $\alpha \to \alpha_D$.

Example 4.1.6. *Recall the pitchfork example with linear multiplicative noise 3.1.2*

$$\mathrm{d}X_t = \left(\alpha X_t - X_t^3\right)\mathrm{d}t + \sigma X_t \circ \mathrm{d}W_t,$$

with cocycle solution

$$\varphi(t,\omega,x) = \frac{x\mathrm{e}^{\alpha t+\sigma W_t(\omega)}}{\left(1 + 2x^2\int_0^t e^{2(\alpha s+\sigma W_s(\omega))}\mathrm{d}s\right)^{1/2}}$$

for all $x \in \mathbb{R}$. Recall the two cases:

(i) For $\alpha \leq 0$, the only invariant measure is given by $\mu_\omega^\alpha = \delta_0$ for all $\omega \in \Omega$.

(ii) For $\alpha > 0$, we have the three invariant Markov measures $\mu_\omega^\alpha = \delta_0$ and $\nu_{\pm,\omega}^\alpha = \delta_{\pm a_\alpha(\omega)}$, where

$$a_\alpha(\omega) = \left(2\int_{-\infty}^0 e^{2(\alpha s+\sigma W_s(\omega))}\right)^{-1/2}.$$

Note that $\mathbb{E}a_\alpha^2 = \alpha$. The Lyapunov exponent λ_1 of the linearized SDE

$$\mathrm{d}v_t = \left(\alpha - 3X_t^2\right)v_t\mathrm{d}t + \sigma v_t \circ \mathrm{d}W_t,$$

with respect to the three measures is

(i) For $\mu_\omega^\alpha = \delta_0$*, we have* $\lambda(\mu^\alpha) = \alpha$*,*

(ii) For $\nu_{\pm,\omega}^\alpha = \delta_{\pm a_\alpha(\omega)}$*, we have* $\lambda(\nu^\alpha) = \alpha - 3\mathbb{E}a_\alpha^2 = -2\alpha$*.*

In summary, we have a D-bifurcation at the reference measure δ_0 *at* $\alpha_D = 0$*.*

[**End of Lecture XII, 29.06.**]

4.2 Random bifurcations (in additive noise SDEs)

Recall from the exercise that in the situation of the pitchfork example with additive noise, the invariant Markov measure is given by $\delta_{A_\alpha(\omega)}$, where $A_\alpha(\omega)$ is the attracting random equilibrium for any $\alpha \in \mathbb{R}$; similarly this is the situation for the Hopf bifurcation with additive noise (and no large shear). In both situations, however, one may still see more subtle bifurcations that cannot be captured by a D-bifurcation, or more generally, loss of topological equivalence. A crucial criterion here is to compare if attraction is *uniform* and if the toplogical equivalence is *uniformly continuous*. This motivates the following definitions and the subsequent observation:

Definition 4.2.1 (Uniform topological equivalence)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space, $\theta : \mathbb{T} \times \Omega \to \Omega$ a metric dynamical system and $(X_1, d_1), (X_2, d_2)$ be Polish spaces. Then, for $\tilde{\mathbb{T}} \in \{\mathbb{T}, \mathbb{T}_0^+\}$, the random dynamical systems $(\theta, \varphi_1 : \tilde{\mathbb{T}} \times \Omega \times X_1 \to X_1)$ and $(\theta, \varphi_2 : \tilde{\mathbb{T}} \times \Omega \times X_2 \to X_2)$ are called *uniformly topologically equivalent* with respect to a random equilibrium $\{a(\omega)\}_{\omega \in \Omega}$ of φ_1 if there exists a conjugacy $h : \Omega \times X_1 \to X_2$ satisfying the following properties:

(i) For almost all $\omega \in \Omega$, the function $x \mapsto h(\omega, x)$ is a homeomorphism from $X_1 \to X_2$.

(ii) The mappings $(\omega, x_1) \mapsto h(\omega, x_1)$ and $(\omega, x_2) \mapsto h^{-1}(\omega, x_2)$ are measurable.

(iii) The random dynamical systems φ_1 and φ_2 are cohomologous, i.e.

$$\varphi_2(t, \omega, h(\omega, x)) = h(\theta_t\omega, \varphi_1(t, \omega, x)) \text{ for all } t \in \tilde{\mathbb{T}}, x \in X_1 \text{ and almost all } \omega \in \Omega.$$

(iv) We have

$$\lim_{\delta \to 0} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in B_\delta(a(\omega))} d_2(h(\omega, x), h(\omega, a(\omega))) = 0$$

and

$$\lim_{\delta \to 0} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in B_\delta(h(\omega, a(\omega)))} d_1(h^{-1}(\omega, x), a(\omega)) = 0.$$

Definition 4.2.2 (Uniform attractivity)**.** In the following, we say that a random equilibrium $a(\omega)$ is *locally uniformly attractive* if there is a $\delta > 0$ such that

$$\lim_{t \to \infty} \operatorname*{ess\,sup}_{\omega \in \Omega} \sup_{x \in B_\delta(a(\omega))} d(\varphi(t, \omega, x), a(\theta_t\omega)) = 0. \tag{4.2.1}$$

Proposition 4.2.3. *Let* $(\Omega, \mathcal{F}, \mathbb{P})$ *be a probability space,* $\theta : \mathbb{T} \times \Omega \to \Omega$ *a metric dynamical system,* $(X_1, d_1), (X_2, d_2)$ *be Polish spaces, and, for* $\tilde{\mathbb{T}} \in \{\mathbb{T}, \mathbb{T}_0^+\}$*, the random dynamical systems*

$(\theta, \varphi_1 : \tilde{\mathbb{T}} \times \Omega \times X_1 \to X_1)$ and $(\theta, \varphi_2 : \tilde{\mathbb{T}} \times \Omega \times X_2 \to X_2)$ be uniformly topologically equivalent with respect to a random equilibrium $\{a(\omega)\}_{\omega\in\Omega}$ of φ_1. Let $h : \Omega \times X_1 \to X_2$ be the conjugacy. Then $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1 if and only if $\{h(\omega, a(\omega))\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_2.

Proof. Suppose that $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1 and let $\eta > 0$. By assumption, there exists a $\gamma > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\gamma(a(\omega))} d_2(h(\omega, x), h(\omega, a(\omega))) \le \eta.$$

Since $\{a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_1, there exists a $\delta > 0$ and a $T > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_1(\varphi(t, \omega, x), a(\theta_t\omega))) \le \frac{\gamma}{2} \quad \text{for all } t \ge T.$$

Hence, for all $t \ge T$, we have

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2(h(\theta_t\omega, \varphi_1(t, \omega, x)), h(\theta_t\omega, a(\theta_t\omega))) \le \eta.$$

This means that, for all $t \ge T$,

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\delta(a(\omega))} d_2(\varphi_2(t, \omega, h(\omega, x))), h(\theta_t\omega, a(\theta_t\omega))) \le \eta.$$

By continuity of $h^{-1}(\omega, \cdot)$, there exists a $\beta > 0$ such that

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\beta(h(\omega, a(\omega)))} d_1(h^{-1}(\omega, x), a(\omega)) \le \frac{\delta}{2}.$$

Finally, this yields all together that for $t \ge T$

$$\operatorname*{ess\,sup}_{\omega\in\Omega} \sup_{x\in B_\beta(h(\omega, a(\omega)))} d_2(\varphi_2(t, \omega, x), h(\theta_t\omega, a(\theta_t\omega))) \le \eta,$$

which means that $\{h(\omega, a(\omega)\}_{\omega\in\Omega}$ is locally uniformly attractive for φ_2. The converse is proved analogously. □

Example 4.2.4. *1. We have seen in the exercises, that for the standard example with additive noise*

$$\mathrm{d}X_t = \left(\alpha X_t - X_t^3\right) \mathrm{d}t + \sigma \,\mathrm{d}W_t,$$

the random equilibrium $a_\alpha(\omega)$ looses uniform attractivity when α crosses 0. Hence, the random dynamical systems for $\alpha < 0$ on the one hand and $\alpha > 0$ on the other are not uniformly topologically equivalent. One may also call this a random bifurcation*.*

2. The other main example from the exercises

$$\begin{aligned} \mathrm{d}x &= (\alpha x - \beta y - (ax - by)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^1\,, \\ \mathrm{d}y &= (\alpha y + \beta x - (bx + ay)(x^2 + y^2))\,\mathrm{d}t + \sigma\,\mathrm{d}W_t^2\,, \end{aligned}$$

with b sufficiently small and all other parameters fixed, in fact, exhibits also this loss of uniform topological equivalence when α crosses 0 (see [11]).

4.3 Bifurcation of a stochastically driven limit cycle

We consider the following model of a stochastically driven limit cycle

$$\begin{aligned} \mathrm{d}y &= -\alpha y\,\mathrm{d}t + \sigma \textstyle\sum_{i=1}^m f_i(\vartheta) \circ \mathrm{d}W_t^i\,, \\ \mathrm{d}\vartheta &= (1 + by)\,\mathrm{d}t\,, \end{aligned} \tag{4.3.1}$$

where $(y, \vartheta) \in \mathbb{R} \times \mathbb{S}^1$ are cylindrical amplitude-phase coordinates, $m \geq 2$ is a natural number, and W_t^i for $i \in \{1, \dots, m\}$ denote independent one-dimensional Brownian motions entering the equation as noise of Stratonovich type.

In the absence of noise ($\sigma = 0$), the ordinary differential equation (4.3.1) has a globally attracting limit cycle at $y = 0$ if $\alpha > 0$. In the presence of noise ($\sigma \neq 0$), the amplitude is driven by phase-dependent noise. The real parameter b induces shear: if $b \neq 0$, the phase velocity $\frac{\mathrm{d}\vartheta}{\mathrm{d}t}$ depends on the amplitude y. The stable limit cycle turns into a random attractor if $\sigma \neq 0$. The main question we address in the following concerns the nature of this random attractor. The crucial quantity is the sign of the first Lyapunov exponent $\lambda_1 = \lambda_1(\alpha, b, \sigma)$ with respect to the invariant measure associated to the random attractor.

The functions $f_i : \mathbb{S}^1 \simeq [0, 1) \to \mathbb{R}$ are assumed to be $C^{2,\delta}$ for some $0 < \delta \leq 1$ to guarantee differentiability of the random dynamical system, and, to facilitate the analysis, we require

$$\sum_{i=1}^m f_i'(\vartheta)^2 = 1 \quad \text{for all } \vartheta \in \mathbb{S}^1\,. \tag{4.3.2}$$

A simple example is given by

$$m = 2, \quad f_1(\vartheta) = \cos(\vartheta), \quad f_2(\vartheta) = \sin(\vartheta)\,. \tag{4.3.3}$$

Using an explicit formula of Furstenberg–Khasminskii type for the top Lyapunov exponent λ_1 derived by Peter Imkeller and Christian Lederer [17], we obtain the following bifurcation result with such choices of the amplitude-phase coupling.

Theorem 4.3.1. *Consider the SDE (4.3.1) with $m \geq 2$ and f_i, $i \in \{1, \dots, m\}$, satisfying assumption (4.3.2). Then there is $c_0 \approx 0.2823$ such that for all $\alpha > 0$ and $b \neq 0$, the number $\sigma_0(\alpha, b) = \frac{\alpha^{3/2}}{c_0^{1/2}|b|} > 0$ is the unique value of σ where the top Lyapunov exponent $\lambda_1(\alpha, b, \sigma)$ of*

(4.3.1) *changes sign:*

$$\lambda_1(\alpha, b, \sigma) \begin{cases} < 0 & \text{if } 0 < \sigma < \sigma_0(\alpha, b)\,, \\ = 0 & \text{if } \sigma = \sigma_0(\alpha, b)\,, \\ > 0 & \text{if } \sigma > \sigma_0(\alpha, b)\,. \end{cases}$$

As long as $b, \sigma \neq 0$, the amplitude variable y can be rescaled so that the shear parameter b becomes equal to 1 and the effective noise amplitude becomes σb. Hence, the above result also holds with the roles of σ and b interchanged. Note that $\sigma_0(\alpha, b)$ is an increasing function of α. If $\sigma = 0$, we clearly have $\lambda_1 = 0$ for all $\alpha > 0$. The case $\alpha = 0$ is obviously not of any interest in our model.

4.3.1 Explicit formula for the Lyapunov exponents

Consider the stochastic differential equation of Stratonovich type (4.3.1) fulfilling condition (4.3.2), and assume that the three parameters fulfill $\alpha > 0$, $\sigma > 0$ and $b \in \mathbb{R}$. Note that the equation reads the same in Itô form according to the Itô–Stratonovich conversion formula.

Since the drift and diffusion coefficients are Lipschitz continuous and satisfy linear growth conditions, the SDE (4.3.1) generates a continuous random dynamical system $(\theta : \mathbb{R} \times \Omega \to \Omega, \varphi : \mathbb{R}_0^+ \times \Omega \times \mathbb{R} \times \mathbb{S}^1 \to \mathbb{R} \times \mathbb{S}^1)$.

In the case of (4.3.1), the variational equation on the tangent space $T_x(\mathbb{R} \times \mathbb{S}^1) \cong \mathbb{R}^2$ takes the form

$$\mathrm{d}v = \underbrace{\begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix}}_{=:A} v \,\mathrm{d}t + \sigma \sum_{i=1}^{m} \underbrace{\begin{pmatrix} 0 & f_i'(\vartheta) \\ 0 & 0 \end{pmatrix}}_{=:B_i} v \circ \mathrm{d}W_t^i\,. \tag{4.3.4}$$

Note that we omit the (t, ω)-dependence of ϑ and B. Observe that the cocycle φ is differentiable in x and recall that its partial derivative $\mathrm{D}\varphi(t, \omega, x)$ with respect to x applied to an initial condition $v_0 \in \mathbb{R}^2$ solves uniquely the variational equation (4.3.4). The random dynamical system (θ, φ) has an ergodic invariant measure ν associated with the stationary measure ρ for (4.3.1) and clearly satisfies the integrability condition

$$\sup_{0 \le t \le 1} \log^+ \|\mathrm{D}\varphi(t, \omega, x)\| \in L^1(\nu).$$

Hence, the MET gives the existence of at least one λ_1 and at most two Lyapunov exponents $\lambda_1 \geq \lambda_2$ for φ via its derivative $\mathrm{D}\varphi$ such that

$$\lim_{t \to \infty} \frac{1}{t} \log \|\mathrm{D}\varphi(t, \omega, x) v\| \in \{\lambda_1, \lambda_2\}\,,$$

for all $v \in \mathbb{R}^2$ and ν-almost all $(\omega, x) \in \Omega \times \mathbb{R} \times \mathbb{S}^1$.

We observe that

$$U_t := \sum_{i=1}^{m} \int_0^t f_i'(\vartheta_s) \mathrm{d}W_s^i \tag{4.3.5}$$

defines a standard Brownian motion under condition (4.3.2). Hence, the variational equation (4.3.4) can be replaced by

$$\mathrm{d}v = \begin{pmatrix} -\alpha & 0 \\ b & 0 \end{pmatrix} v \,\mathrm{d}t + \begin{pmatrix} 0 & \sigma \\ 0 & 0 \end{pmatrix} v \circ \mathrm{d}U_t\,, \tag{4.3.6}$$

and the Lyapunov exponents for φ can be computed from this equation.

The following theorem is now a direct corollary of a result by Imkeller and Lederer [17, Theorem 3], using the Furstenberg-Khasminskii formula (2.5.30).

Theorem 4.3.2. *Consider the stochastic differential equation* (4.3.1) *with* $m \geq 2$ *and* f_i*,* $i \in \{1,\dots,m\}$*, satisfying assumption* (4.3.2)*. Then the two Lyapunov exponents are given by*

$$\lambda_1(\alpha,b,\sigma) = -\frac{\alpha}{2} + \frac{|b\sigma|}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,, \tag{4.3.7}$$

$$\lambda_2(\alpha,b,\sigma) = -\frac{\alpha}{2} - \frac{|b\sigma|}{2}\int_0^\infty v\; m_{\sigma,b,\alpha}(v)\,\mathrm{d}v\,, \tag{4.3.8}$$

where

$$m_{\sigma,b,\alpha}(v) = \frac{\frac{1}{\sqrt{v}}\exp\left(-\frac{|b\sigma|}{6}v^3 + \frac{\alpha^2}{2|b\sigma|}v\right)}{\int_0^\infty \frac{1}{\sqrt{u}}\exp\left(-\frac{|b\sigma|}{6}u^3 + \frac{\alpha^2}{2|b\sigma|}u\right)\mathrm{d}u}. \tag{4.3.9}$$

Proof. Replacing $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ by $\hat{v} = \begin{pmatrix} v_2 \\ \frac{v_1}{\sigma} \end{pmatrix}$ leaves the Lyapunov exponents invariant and transforms (4.3.6) into the equation

$$\mathrm{d}v = \begin{pmatrix} 0 & \sigma b \\ 0 & -\alpha \end{pmatrix} v\,\mathrm{d}t + \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} v \circ \mathrm{d}W_t^1\,. \tag{4.3.10}$$

The matrices in this equation satisfy the assumptions of [17, Theorem 3] which gives the formulas (4.3.7) and (4.3.8). □

4.3.2 Bifurcation from negative to positive top Lyapunov exponent

We now use Theorem 4.3.2 to prove Theorem 4.3.1, which asserts that there is a bifurcation from negative to positive Lyapunov exponent for the stochastic differential equation (4.3.1).

Proof of Theorem 4.3.1. We fix $\alpha > 0$ and $b \neq 0$. Introducing the change of variables $v = \frac{\alpha}{|b\sigma|}u$ in (4.3.9), we obtain

$$\lambda_1(\alpha,b,\sigma) = \frac{\alpha}{2}\left(\int_0^\infty u\;\tilde{m}_{\sigma,b,\alpha}(u)\,\mathrm{d}u - 1\right), \tag{4.3.11}$$

where

$$\tilde{m}_{\sigma,b,\alpha}(u) = \frac{\frac{1}{\sqrt{u}}\exp\left(-\frac{\alpha^3}{\sigma^2 b^2}\left[\frac{1}{6}u^3 - \frac{1}{2}u\right]\right)}{\int_0^\infty \frac{1}{\sqrt{w}}\exp\left(-\frac{\alpha^3}{\sigma^2 b^2}\left[\frac{1}{6}w^3 - \frac{1}{2}w\right]\right)\mathrm{d}w}\,.$$

Defining $c := \frac{\alpha^3}{\sigma^2 b^2}$, we observe that $\lambda_1(\alpha, b, \sigma)$ has the same sign as the function $G : (0, \infty) \to \mathbb{R}$ given by

$$G(c) := \int_0^\infty \left(\sqrt{u} - \frac{1}{\sqrt{u}}\right) \exp\left(-c\left[\frac{1}{6}u^3 - \frac{1}{2}u\right]\right) \mathrm{d}u\,. \tag{4.3.12}$$

Using dominated convergence, we may interchange the order of differentiation and integration and consider

$$G'(c) = \int_0^\infty h_1(u) h_2(u) \exp\left(c\, h_2(u)\right) \mathrm{d}u\,,\ h_1(u) = \sqrt{u} - \frac{1}{\sqrt{u}}\,,\ h_2(u) = -\frac{1}{6}u^3 + \frac{1}{2}u\,.$$

Note that $h_1 h_2$, and thereby the integrand, has positive sign on the interval $(1, \sqrt{3})$ and negative sign on $(0, 1)$ and $(\sqrt{3}, \infty)$. Moreover, $|h_1(1-\delta)| > h_1(1+\delta)$ and $h_2(1-\delta) > h_2(1+\delta)$ for all $\delta \in (0, \sqrt{3}-1)$ so that

$$G'(c) < \int_{2-\sqrt{3}}^{\sqrt{3}} h_1(u) h_2(u) \exp\left(c\, h_2(u)\right) \mathrm{d}u < 0 \quad \text{for all } c \in (0, \infty)\,.$$

Hence, G is strictly decreasing. Furthermore, we observe that $G(c) \to \infty$ as $c \searrow 0$ (using monotone convergence on $[\sqrt{3}, \infty)$) and that $G(c) \to -\infty$ as $c \to \infty$ (using similar arguments as for G' and monotone convergence on $(0, 2-\sqrt{3})$).

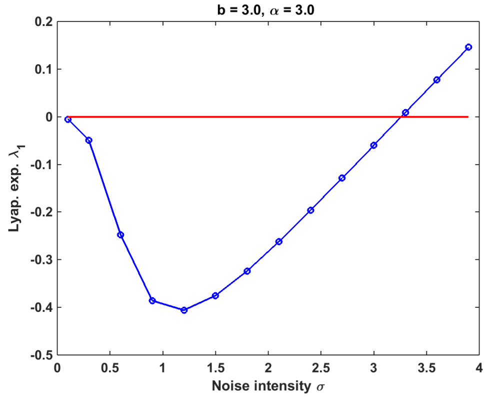
Combining these observations, we conclude that there is a unique c_0 such that $G(c_0) = 0$, $G(c) > 0$ for all $c \in (0, c_0)$ and $G(c) < 0$ for all $c \in (c_0, \infty)$. This proves the claim with $\sigma_0(\alpha, b) = \frac{\alpha^{3/2}}{c_0^{1/2}|b|}$. Numerical integration gives $c_0 \approx 0.2823$. □

Note that, as explained already above, the same result holds if we interchange the roles of σ and b. This can be seen also directly from the proof.

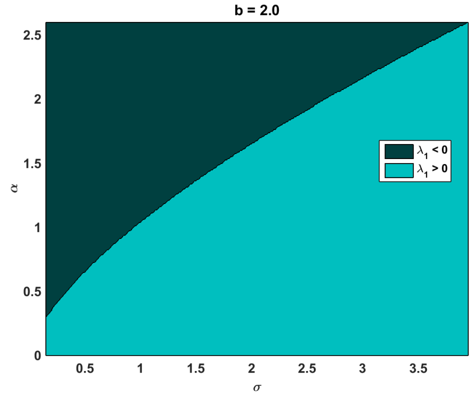
Remark 4.3.3. The random dynamical system induced by (4.3.1) has a random set attractor $\{\tilde{A}(\omega)\}_{\omega\in\Omega}$ for all parameter values, as can be seen similarly to [18]. Furthermore, the random dynamical system possesses an ergodic invariant Markov measure μ which is associated to the unique invariant measure (also called stationary measure) for the corresponding Markov semigroup. The disintegrations μ_ω of the ergodic invariant measure μ are supported on subsets of the fibers $\tilde{A}(\omega)$, i.e. $A(\omega) := \operatorname{supp}(\mu_\omega) \subset \tilde{A}(\omega)$. In fact, the measurable random compact set $\{A(\omega)\}_{\omega\in\Omega}$ is a minimal (weak) random point attractor of (4.3.1), see Proposition 3.2.8.

The fact that $\{A(\omega)\}_{\omega\in\Omega}$ is a singleton almost surely if $\lambda_1 < 0$, follows from a slightly modified reasoning alongside Theorem 3.2.9 and its proof. In the case of $\lambda_1 > 0$, we deduce that μ_ω is atomless almost surely as in the proof of Theorem 3.3.1. Hence, Theorem 4.3.1 implies the bifurcation from an attractive random equilibrium to an atomless random point attractor (also called random strange attractor).

In Figure 4.1(a), we show the top Lyapunov exponent as a function of σ for fixed b and α according to formula (4.3.7), illustrating the bifurcation of the sign of λ_1. Figure 4.1(b) displays the (σ, α)-parameter space being separated by the curve $\{(\sigma_0(\alpha, b), \alpha)\}$ for fixed b.



(a)



(b)

Figure 4.1: In Figure 1(a) the top Lyapunov exponent λ_1, calculated according to (4.3.7) using numerical integration, is shown as a function of σ for fixed b and α. Figure 1(b) shows the areas of positive and negative λ_1 in the (σ, α)-parameter space being separated by the curve $\{(\sigma_0(\alpha, 3), \alpha)\}$ as a function of α for fixed $b = 3$, using the formula for σ_0 in Theorem 4.3.1.

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