Ornstein–Uhlenbeck operators and semigroups

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Bogachev, Russian Math. Surveys, 2018 (doi 10.1070/RM9812), Gaussian measures, AMS, 1997, Differentiable measures and the Malliavin calculus, AMS, 2010

The Laplace operator

$$\Delta\varphi(x) = \sum_{i=1}^{d} \partial_{x_i}^2 \varphi(x)$$

The Ornstein–Uhlenbeck operator

$$L\varphi(x) = \Delta\varphi(x) - \langle x, \nabla\varphi(x)\rangle$$

Equations:

$$\partial_t u = \Delta u,\ \partial_t u = Lu,\ u(x,0) = f(x).$$

The heat semigroup

$$P_t f(x) = \int f(x-y) \frac{1}{\sqrt{4\pi t}^d} \exp\Big(-\frac{|y|^2}{4t}\Big)\, dy,$$

The Ornstein–Uhlenbeck semigroup

$$T_t f(x) = \int_{\mathbb{R}^d} f\left(e^{-t}x - \sqrt{1-e^{-2t}}\, y\right) \gamma(dy),$$

where γ is the standard Gaussian measure on $\mathbb{R}^d$ with density

$$\varrho(x) = \frac{1}{(2\pi)^{d/2}} e^{-|x|^2/2}$$

Important contributors:
Laplace, Gauss, Chebyshev, Hermite, Mehler, Bachelier, Smoluchowski, Wiener, Ornstein, Uhlenbeck, Hille, Doob, Nelson, Gross...

QUESTION: Who did coin the terms "Ornstein–Uhlenbeck semigroup" and "Ornstein–Uhlenbeck operator"?

Operator semigroups:

$$T_{t+s} = T_t \circ T_s, \quad T_0 = I, \; s, t \geq 0.$$

Strongly continuous semigroups on a Banach space X:

$$\lim_{t \geq 0} T_t x = x \; \forall \, x \in X.$$

What are suitable X for P_t and T_t?
T_t takes C_b into C_b, L^∞ into L^∞, but not strongly continuous:

Take a bounded Lipschitz function f for which $f(n) = 1$ and $f(e^{-1/n}n) = 0$. It exists, since $(1 - e^{-1/n})n \to 1$. Then

$$
\begin{aligned}
&|T_{1/n}f(x) - f(e^{-1/n}x)| \\
&\leq \int \left| f\left(e^{-1/n}x - \sqrt{1 - e^{-2/n}}\, y\right) - f(e^{-1/n}x) \right| \gamma(dy) \\
&\leq C\sqrt{1 - e^{-2/n}},
\end{aligned}
$$

so $\| T_{1/n}f - f \|_\infty$ cannot tend to zero because $f(e^{-1/n}n) - f(n) = 1$.

$L^p(\gamma)$ with $1 \leq p < \infty$ are suitable

Theorem 1. $\{T_t\}$ is strongly continuous on $L^p(\gamma)$ with $p < \infty$ and $\|T_t f\|_{L^p(\gamma)} \leq \|f\|_{L^p(\gamma)}$.

The semigroup property: exercise in calculus and change of variable

Strong continuity: the bound follows by Hölder; for $f \in C_0^\infty$ the norm continuity is trivial; for any f follows by approximation.

Remark. Why $L^p(\gamma)$, not $L^p(\mathbb{R}^d)$?
In some respects $L^p(\mathbb{R}^d)$ is also fine:
Since T_t takes L^∞ to L^∞ and also takes $L^1(\mathbb{R}^d)$ to $L^1(\mathbb{R}^d)$ due to

$$\int_{\mathbb{R}^d} |T_t f(x)| \, dx$$

$$\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(e^{-t}x + \sqrt{1 - e^{-2t}}\, y)| \, \gamma(dy) \, dx$$

$$= e^{dt} \int_{\mathbb{R}^d} |f(u)| \, du,$$

the case $p > 1$ follows by the interpolation theorem.
On $L^p(\mathbb{R}^d)$ with $p < \infty$ the semigroup $\{T_t\}$ is also continuous.

Something IMPORTANT:

the measure γ is invariant for T_t, i.e.,

$$\int T_t f \, d\gamma = \int f \, d\gamma \quad \forall f \in L^1(\gamma),$$

and, with the integral of f denoted by $I(f)$,

$$\lim_{t\to\infty} \| T_t f - I(f) \|_p = 0 \quad \forall f \in L^p(\gamma), \quad p \in [1, \infty).$$

On $L^2(\gamma)$ the operators T_t are self-adjoint and non-negative in the sense of quadratic forms, and on $L^p(\gamma)$ they are non-negative in the sense of ordered spaces, i.e., take non-negative functions to non-negative functions.

Some other useful relations for L and T_t: in place of the usual integration by parts formula

$$\int f \Delta g \, dx = - \int \langle \nabla f, \nabla g \rangle \, dx$$

for smooth functions with compact support one has

$$\int fLg \, d\gamma = - \int \langle \nabla f, \nabla g \rangle \, d\gamma$$

Next,

$$\nabla T_t f = e^{-t} T_t \nabla f.$$

From Jensen's inequality for a convex function V:

$$V(T_t f) \leq T_t(V(f)).$$

In particular, for $f > 0$ we have

$$T_t \ln f \leq \ln T_t f \quad \text{and} \quad T_t(f \ln f) \geq T_t f \ln T_t f,$$

From the general theory of continuous operator semigroups on Banach spaces: for every $p \in [1, \infty)$ the set

$$D_p(L) := \left\{ f \in L^p(\gamma) \colon \lim_{t \to 0} \frac{1}{t}(T_t f - f) \text{ exists in } L^p(\gamma) \right\}$$

is a dense linear subspace of $L^p(\gamma)$, and the linear operator with domain $D_p(L)$ given by

$$Lf := \lim_{t \to 0} \frac{1}{t}(T_t f - f)$$

is closed, that is, has a closed graph: if $f_n \in D_p(L)$, $f_n \to f$, and $Lf_n \to g$ in $L^p(\gamma)$, then $f \in D_p(L)$ and $Lf = g$.

This operator is called the generator of the semigroup $\{T_t\}$. In the case of the Ornstein–Uhlenbeck semigroup, L is called the Ornstein–Uhlenbeck operator. It is convenient to write the operators T_t in the form $T_t = \exp(tL)$, with L on $L^2(\gamma)$ having a non-positive quadratic form, that is, to indicate that the corresponding operator exponential $\exp(tL)$ coincides with T_t.

HOW TO FIND L EXPLICITLY?

On some functions f, say, from C_0^∞, this is easy:

$$
\begin{aligned}
& f\left(e^{-t}x - \sqrt{1-e^{-2t}}\, y\right) - f(x) \\
& = \int_0^t f'\left(e^{-s}x - \sqrt{1-e^{-2s}}\, y\right) \\
& \times \left(-e^{-s}x - e^{-2s}(1-e^{-2s})^{-1/2} y\right) ds.
\end{aligned}
$$

After integration in y with respect to γ we obtain two terms. The first term multiplied by t^{-1} tends in $L^p(\gamma)$ to $-xf'(x)$ as $t \longrightarrow 0$ by the Lebesgue theorem. The second term is transformed by integration by parts with respect to y into

$$
\int_0^t \int f''\left(e^{-s}x - \sqrt{1-e^{-2s}}\, y\right) e^{-2s}\, ds\, \gamma(dy),
$$

which with the factor t^{-1} tends to $f''(x)$ in $L^p(\gamma)$ as $t \longrightarrow 0$.

Thus, Lf is the action of the Ornstein–Uhlenbeck operator.

But what is the exact domain of L on $L^p(\gamma)$?

A simple case is $p = 2$, where Chebyshev–Hermite polynomials can be used (called Hermite polynomials for brevity) defined by

$$H_0 = 1, \qquad H_k(t) = \frac{(-1)^k}{\sqrt{k!}} e^{t^2/2} \frac{d^k}{dt^k} e^{-t^2/2}, \quad k \geq 1.$$

The crucial fact is that $\{H_k\}$ is an orthonormal basis in $L^2(\gamma)$ and

$$T_t H_k = e^{-kt} H_k, \quad LH_k = -kH_k.$$

In $\mathbb{R}^d$, where k is a multi-index $k = (k_1, \ldots, k_d)$,

$$T_t H_{k_1,\ldots,k_d} = e^{-(k_1+\cdots+k_d)t} H_{k_1,\ldots,k_d}.$$

Theorem 2. The domain of L in $L^2(\gamma)$ for $d = 1$ is

$$D_2(L) = \left\{ f = \sum_{k=0}^{\infty} c_k H_k \colon \sum_{k=0}^{\infty} k^2 |c_k|^2 < \infty \right\},$$

and

$$Lf = -\sum_{k=0}^{\infty} k c_k H_k.$$

Similarly in the multidimensional case.

The proof is straightforward:

$$\frac{T_t f - f}{t} = \sum_k \frac{e^{-kt} - 1}{t} c_k H_k, \;\; f = \sum_k c_k H_k.$$

If $\sum_{k=0}^{\infty} k^2 |c_k|^2 < \infty$, then letting $g = -\sum_k k c_k H_k$, we have

$$\|(T_t f - f)/t - g\|^2 = \sum_k \left| \frac{e^{-kt} - 1}{t} + k \right|^2 |c_k|^2 \to 0$$

as $t \to 0$. Conversely, if there is a limit of $(T_t f - f)/t$ in $L^2(\gamma)$, its coordinates in $\{H_k\}$ must be $-kc_k$, so the series of $k^2 |c_k|^2$ must converge.
It is verified directly that LH_k is indeed the action of the Ornstein–Uhlenbeck operator.

How useful is this description? If $f \in C_0^\infty$, then $f \in D_2(L)$ without this theorem, but is it seen from the theorem?

The theorem shows that $H_k \in D_2(L)$, hence all polynomials are in $D_2(L)$, but how can we check that f is in $D_2(L)$ without expansions?

Sobolev classes: $W^{p,k}(\mathbb{R}^d)$ consists of $f \in L^p(\mathbb{R}^d)$ such that the distributional derivatives of f up to order k are functions from $L^p(\mathbb{R}^d)$, where a locally integrable g is $\partial_{x_i} f$ in the sense of distributions if

$$\int \partial_{x_i}\varphi \, f \, dx = -\int \varphi g \, dx$$

for all $\varphi \in C_0^\infty$.

DEFINITION. $W^{p,k}_{loc}(\mathbb{R}^d)$ consists of functions f such that $\zeta f \in W^{p,k}(\mathbb{R}^d)$ for all $\zeta \in C_0^\infty$.

Theorem 3. $W^{p,k}(\mathbb{R}^d)$ coincides with the completion of C_0^∞ with respect to the Sobolev norm

$$\|f\|_{W^{p,k}(\mathbb{R}^d)} = \|f\|_{L^p(\mathbb{R}^d)} + \sum_{m \le k} \|\partial_{x_{i_1}} \cdots \partial_{x_{i_m}} f\|_{L^p(\mathbb{R}^d)}$$

Theorem 4. Let $p > 1$. Then $W^{p,2}(\mathbb{R}^d)$ consists of $f \in L^p(\mathbb{R}^d)$ such that Δf in the sense of distributions is represented by an element of $L^p(\mathbb{R}^d)$.

FALSE for $p = 1$

GAUSSIAN ANALOGS:

DEFINITION. $W^{p,k}(\gamma)$ is the completion of C_0^∞ with respect to the Sobolev norm

$$\|f\|_{W^{p,k}(\gamma)} = \|f\|_{L^p(\gamma)} + \sum_{m\le k} \|\partial_{x_{i_1}} \partial_{x_{i_m}} f\|_{L^p(\gamma)}$$

Theorem 5. $W^{p,k}(\gamma)$ consists of all $f \in W^{p,k}_{loc}(\mathbb{R}^d)$ such that $\|f\|_{W^{p,k}(\gamma)} < \infty$.

Let $p > 1$. Then $W^{p,2}(\gamma)$ consists of all $f \in W^{p,2}_{loc}(\mathbb{R}^d)$ such that $f \in L^p(\gamma)$ and $\Delta f - \langle x, \nabla f\rangle \in L^p(\gamma)$.

Theorem 6. $W^{p,2}(\gamma) = D_p(L)$

The rth-order derivative of a function f will be denoted by $D^r f$. The gradient ∇f will be denoted also by Df for uniformity (however, sometimes one writes $\nabla^r f$ instead of $D^r f$). We recall that the Hilbert–Schmidt norm of the derivative $D^r f(x)$ is defined by

$$\|D^r f(x)\|_{\mathcal{H}_r} := \Bigg(\sum_{1 \leqslant i_j \leqslant d} |\partial_{x_{i_1}} \cdots \partial_{x_{i_r}} f(x)|^2 \Bigg)^{1/2}.$$

Theorem 7. Let If $p \in (1, \infty)$ and $r \in \mathbb{N}$, then there are numbers $m_{p,r}$ and $M_{p,r}$ independent of d such that

$$m_{p,r}\|D^r f\|_{L^p(\gamma,\mathcal{H}_r)} \le \|(I-L)^{r/2} f\|_{L^p(\gamma)}$$

$$\le M_{p,r}\left[\|D^r f\|_{L^p(\gamma,\mathcal{H}_r)} + \|f\|_{L^p(\gamma)}\right].$$

In particular,

$$m_{p,2}\|D^2 f\|_{L^p(\gamma,\mathcal{H}_2)} \le \|(I-L) f\|_{L^p(\gamma)}$$

$$\le M_{p,2}\left[\|D^2 f\|_{L^p(\gamma,\mathcal{H}_2)} + \|f\|_{L^p(\gamma)}\right].$$

The operator $(I - L)^{-1}$ is a self-adjoint contraction on $L^2(\gamma)$, hence $(I - L)^{-r/2}$ is also for any $r > 0$. One can show that $(I - L)^{-r/2}$ is a contraction and injection on each $L^p(\gamma)$.
Set

$$H^{p,r}(\gamma) := (I - L)^{-r/2}(L^p(\gamma))$$

Theorem 8. If $p > 1$, $r \in \mathbb{N}$, then

$$W^{p,r}(\gamma) = H^{p,r}(\gamma)$$

INFINITE DIMENSIONAL EXTENSIONS

γ_d is the product of d copies of γ_1
On the space $\mathbb{R}^\infty$ of all real sequences $x = (x_1, x_2, \ldots)$ (the countable power of $\mathbb{R}$) we define γ as the countable power of γ_1.
This means that on cylindrical sets

$$C = \{x \colon (x_1, \ldots, x_d) \in B\}$$

the value of γ is $\gamma_d(B)$. Then γ extends to the σ-algebra generated by such cylinders.
A very special measure? No.

A general centered Gaussian measure γ_0 on a separable Banach space X is a Borel probability measure on X such that every continuous linear functional on X is a centered Gaussian random variable on (X, γ).

If γ_0 is not concentrated on a finite-dimensional space, then γ_0 is linearly isomorphic to γ in the following sense: one can find Borel linear subspaces $X_0 \subset X$ and $E \subset \mathbb{R}^\infty$ and a Borel linear operator j that takes E one-to-one onto X_0 and the image of γ is γ_0.

Now T_t on $L^p(\gamma)$ is defined by the same expression and gives a strongly continuous semigroup.
$W^{p,k}(\gamma)$ is the completion with respect to the Sobolev norm of the union of $W^{p,k}(\gamma_d)$ over all d.
Again L is defined on $D_p(L)$ as the generator of $\{T_t\}$ and $(I-L)^{-r/2}$ extends from $L^2(\gamma)$ to all $L^p(\gamma)$ as an injective contraction.
As above, $H^{p,r}(\gamma) :=:= (I-L)^{-r/2}(L^p(\gamma))$ and $H^{p,r}(\gamma) = W^{p,r}(\gamma)$ for natural r.

L on functions in finitely many variables is the same.
BUT: new phenomena for functions of infinitely many variable.
EXAMPLE: $f(x) = \sum_i c_i(x_i^2 - 1)$, $\sum_i c_i^2 < \infty$, the series converges in $L^2(\gamma)$ (in all $L^p(\gamma)$).
$f_n(x) = \sum_{i \le n} c_i(x_i^2 - 1)$ converge to f,

$$Lf_n(x) = \sum_{i \le n} (\partial_{x_i}^2 f_n(x) - x_i \partial_{x_i} f_n(x))$$

$$= \sum_{i \le n} 2c_i(1 - x_i^2) = 2f_n(x),$$

so f is in $W^{2,2}(\gamma)$, actually, in all $W^{p,k}(\gamma)$, but the series of $\partial_{x_i}^2 f = 2c_i$ and $x_i \partial_{x_i} f = c_i x_i^2$ do not converge separately if $c_i = 1/i$.

REMARK on $p = 1$:

The set $D_1(L)$ (domain of L on $L^1(\gamma)$ consists of all $f \in L^1(\gamma)$ such that the distribution $\Delta f - \langle x, \nabla f \rangle$ is given by a function in $L^1(\gamma)$. Moreover, $D_1(L)$ strictly contains $W^{1,2}(\gamma)$.

Sobolev embeddings:

$f \in W^{1,1}(\mathbb{R}^d) \Rightarrow f \in L^{d/(d-1)}(\mathbb{R}^d)$

$f \in W^{2,1}(\mathbb{R}^d) \Rightarrow f \in L^{2d/(d-2)}(\mathbb{R}^d)$, $d \geq 2$

$f \in W^{p,1}(\mathbb{R}^d)$, $p > d \Rightarrow f \in L^{\infty}(\mathbb{R}^d)$

ALL IS FALSE for $W^{p,k}(\gamma)$
INSTEAD:

Logarithmic Sobolev inequality:

Theorem 9. $f \in W^{2,1}(\gamma) \Rightarrow f^2 \log |f| \in L^1(\gamma)$ and

$$\int f^2 \log |f| \, d\gamma \leq \int |\nabla f|^2 \, d\gamma$$

$$+ \frac{1}{2}\left(\int f^2 \, d\gamma\right) \log\left(\int f^2 \, d\gamma\right).$$

So if $\|f\|_{L^2(\gamma)} = 1$, then

$$\int f^2 \log |f| \, d\gamma \leq \int |\nabla f|^2 \, d\gamma$$

In infinite dimension:
$H = l^2$ the Cameron–Martin space of γ is the set of all $h \in \mathbb{R}^\infty$ such that the shift γ_h defined by

$$\gamma_h(B) = \gamma(B - h)$$

is equivalent to γ

$$\gamma(H) = 0$$

but H is very important for γ

functions $f \in W^{2,1}(\gamma)$ possess H-valued gradients $\nabla f \in W^{2,1}(\gamma, H)$:
if f_n smooth on $\mathbb{R}^n$ converge to f in $L^2(\gamma)$ and form a Cauchy sequence in $W^{2,1}(\gamma)$, we have

$$\int |\nabla f_m - \nabla f_k|^2 \, d\gamma \to 0$$

as $m, k \to \infty$, but $L^2(\gamma, H)$ of H-valued mappings is also complete.
Any $f \in W^{2,1}(\gamma)$ has a version such that $\partial_{x_i} f$ exist γ-almost everywhere. Then

$$\nabla f = (\partial_{x_i} f)_{i=1}^{\infty}, \ |\nabla f|^2 = \sum_i |\partial_{x_i} f|^2$$

Let $f \geq 0$ and f and $f \log f$ are γ-integrable where $f(x) \ln f(x) := 0$ if $f(x) = 0$
The entropy:

$$\mathrm{Ent}_\gamma(f) := \int f \log f \, d\gamma - \int f \, d\gamma \ \log \int f \, d\gamma.$$

By Jensen's inequality for the convex function $t \ln t$ we have $\mathrm{Ent}_\gamma(f) \geq 0$.
For nice f (say, $f \geq c > 0$ of class C_b^∞) one has

$$\mathrm{Ent}_\gamma(f) = -\int_0^\infty \frac{d}{dt}\left(\int T_t f \log T_t f \, d\gamma\right) dt.$$

Under the integral sign

$$\frac{d}{dt}\int T_t f \log T_t f \, d\gamma = \int LT_t f \log T_t f \, d\gamma + \int LT_t f \, d\gamma$$

$$= -\int \frac{|\nabla T_t f|^2}{T_t f} \, d\gamma,$$

where we use

$$\int L\psi \, \varphi \, d\gamma = -\int \langle \nabla \psi, \nabla \varphi \rangle \, d\gamma,$$

the symmetry of L in $L^2(\gamma)$ and the equality $L1 = 0$. Hence

$$\mathrm{Ent}_\gamma(f) = \int_0^\infty \int \frac{|\nabla T_t f|^2}{T_t f} \, d\gamma \, dt.$$

Use this to prove log-Sobolev:
first, $\nabla T_t f = e^{-t} T_t \nabla f$, next, by the Cauchy–Bunyakovskii

$$|T_t \nabla f|^2 \leq T_t f T_t \left(\frac{|\nabla f|^2}{f} \right),$$

so the previous representation gives

$$\mathrm{Ent}_\gamma(f) \leq \int_0^\infty e^{-2t} \left(\int T_t \left(\frac{|\nabla f|^2}{f} \right) d\gamma \right) dt$$

$$= \frac{1}{2} \int \frac{|\nabla f|^2}{f} \, d\gamma.$$

Now take f^2 in place of f.

For $p \geq 2$:

$$\int |f|^p \log\left(\frac{|f|}{\|f\|_p}\right) d\gamma \leq \frac{p}{2} \int |f|^{p-2} |\nabla f|^2 \, d\gamma,$$

where for $f \in W^{p,2}(\gamma)$ such that $f \geq$ the right-hand side equals the integral of the function $\dfrac{p}{2(p-1)} f^{p-1} Lf$.

The logarithmic Sobolev inequality is equivalent to the hypercontractivity property.

Theorem 10. The Ornstein–Uhlenbeck semigroup $\{T_t\}$ is hypercontractive: for all $p > 1$ and $q > 1$

$$\|T_t f\|_q \leq \|f\|_p$$

for all $t > 0$ such that $e^{2t} \geq (q-1)/(p-1)$.

Theorem 11. Let $p > 1$ and $f \in L^p(\gamma)$. Then $T_t f \in W^{p,n}(\gamma)$ for all $t > 0$ and $n \geq 1$, the function $h \mapsto T_t f(x+h)$ is infinitely Fréchet differentiable on H.

Moreover, $T_t f \in W^{q,n}(\gamma)$ for all $q < 1 + (p-1)e^{2t}$. Therefore, for fixed $q > 1$ and $n \geq 1$ the inclusion $T_t f \in W^{q,n}(\gamma)$ holds for all sufficiently large t.

Embeddings for $W^{1,1}(\gamma)$:

Theorem 12. There is a number C such that for all d and $f \in W^{1,1}(\gamma)$

$$\int |x_i f(x)| \, \gamma_d(dx) \le C\|f\|_{1,1}$$

for all i.

Theorem 13. If $f \in W^{1,1}(\gamma)$, then $f\sqrt{\log|f|} \in L^1(\gamma)$.
Moreover,

$$\|f\|_{L\sqrt{\log L}} \le C\|f\|_{W^{1,1}(\gamma)}$$

Orlicz norm:

$$\|f\|_\Phi = \inf\left\{\lambda > 0\colon \int \Phi(|f|/\lambda)\, d\gamma \le 1\right\}$$

Theorem 14. If $f\sqrt{\log|f|} \in L^1(\gamma)$, then $T_t f \in W^{1,1}(\gamma)$ for all $t > 0$.
Moreover, for every $t > 0$, on bounded functions

$$\int |\nabla T_t f| \, d\gamma \leq C(t) \|f\|_{L\sqrt{\log L}},$$

where

$$C(t) = 2\frac{e^{-t}}{(1 - e^{-2t})^{1/2}}.$$

Hence extends to $L\sqrt{\log L}$.

Theorem 15. There is a constant C such that for all $d \geq 1$, $r > 1$, and $t > 0$ the inequality

$$\gamma_d(x\colon\, T_t f(x) > r) \leq C \max\left\{1, \frac{1}{t}\right\} \frac{1}{r\sqrt{\log r}}$$

holds for every probability density f with respect to the measure γ_d. Hence this also holds in infinite dimension.

If $f \in L^1(\gamma)$, then

$$T_t f \longrightarrow f \quad \text{in } L^1(\gamma) \quad \text{as } t \longrightarrow 0.$$

Is it true that

$$T_t f(x) \longrightarrow f(x) \quad \text{as } t \longrightarrow 0$$

almost everywhere?
YES in $\mathbb{R}^d$, OPEN in infinite dimension

MAXIMAL FUNCTION:

$$Mf(x) := \sup_{t>0} |T_t f(x)|.$$

Theorem 16. For every d, there is c_d such that

$$\gamma_d\Big(x\colon Mf(x) > R\Big) \le c_d R^{-1}$$

for all $R > 1$ and f with $\|f\|_{L^1(\gamma)} = 1$.

OPEN: can we take c_d independent of d?

Theorem 17. If $f \in L^p(\gamma)$ with $p > 1$, then $Mf \in L^p(\gamma)$ and

$$\|Mf\|_{L^p(\gamma)} \leq C(p)\|f\|_{L^p(\gamma)}.$$

Moreover, there is a version of $T_t f$ such that $T_t f(x) \to f(x)$ as $t \to 0$ for almost every x.

REMARK. One has to be careful with versions in infinite dimension: given a Borel set $B \subset \mathbb{R}^\infty$, the usual version

$$T_t I_B(0) = \gamma((1 - e^{-2t})^{-1/2} B)$$

is not always continuous.

A competing maximal function

$$M_d f(x) := \sup_{r>0} \frac{1}{\gamma_d(B(x,r))} \int_{B(x,r)} |f(y)|\, \gamma_d(dy),$$

where $B(x,r)$ is the ball of radius r centered at x. Then

$$\gamma\big(x\colon M_d f(x) > R\big) \le C_d R^{-1}$$

with the minimal possible C_d for all f with $\|f\|_{L^1(\gamma)} = 1$. Then $C_d \to \infty$ as $d \to \infty$.

Wang's Harnack inequality:

Theorem 18. $f \in L^p(\gamma_d)$. If $p > 1$, then

$$|T_t f(y)|^p \leq T_t|f|^p(x) \exp\left(\frac{1}{2}\frac{p}{p-1}\frac{|x-y|^2}{e^{2t}-1}\right).$$

If $0 < p < 1$ and $f \geq 0$, then

$$(T_t f(y))^p \geq T_t f^p(x) \exp\left(\frac{1}{2}\frac{p}{p-1}\frac{|x-y|^2}{e^{2t}-1}\right).$$

PERTURBATIONS OF THE O-U OPERATOR/SEMIGROUP:

$$L_v f(x) = Lf(x) + \langle \nabla f(x), v(x) \rangle$$

$v\colon \mathbb{R}^d \to \mathbb{R}^d$ or $v\colon \mathbb{R}^\infty \to H = l^2$.

Problems: associated semigroups, Kolmogorov equations, etc.

EXAMPLE.

$$L^* \gamma = 0$$

in the following sense:

$$\int Lf \, d\gamma = 0$$

for all $f \in C_0^\infty$ on $\mathbb{R}^d$ and similarly on $\mathbb{R}^\infty$.

Theorem 19. There are no other probability measures satisfying this equation.

The equation

$$L_v^* \mu = 0$$

is understood similarly:

$$\int L_v f \, d\mu = 0$$

for all smooth f in finitely many variables, where it is also required that $v = (v_i)$ with $v_i \in L^1(\mu)$.

Theorem 20. If $|v| \in L^1(\mu)$, then μ is absolutely continuous with respect to γ.

OPEN: Is it true that for $f = d\mu/d\gamma$ one has $f\sqrt{\log f} \in L^1(\gamma)$?